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SOLUTION OF ELASTIC AND ELASTO-PLASTIC PROBLEMS BY THE METHOD OF LINES

Alexander Mendelson and Javed Alam

Case Western Reserve University

Cleveland, Ohio 44106

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SOLUTION OF ELASTIC AND ELASTO-PLASTIC PROBLEMS BY THE METHOD OF LINES

Abstract

by

ALEXANDER MENDELSON and JAVED ALAM

An improved formulation of the method of lines (MOL) is presented. The line method lies midway between completely analytical methods and completely discrete methods such as finite differences. The five-point finite difference formulas are introduced to reduce a set of coupled partial differential equations into a set of simultaneous ordinary differential equations. The resulting ordinary differential equations are solved by a recurrence relation method, which is found to be very suitable in solving two-point boundary value problems.

The formulation is further extended to include the small scale plasticity effect. The Von-Mises criterion is used for yielding and isotropic hardening rule is followed to decide for subsequent yielding. Prantl-Reuss equations are employed as constitutive relations. The resulting nonlinear equations are solved by using the method known as successive elastic solutions.

For two specific geometries namely the edge notch specimen and the compact tension specimen, the complete field solutions for stresses and strains are obtained.

A numerical experimentation is carried out to establish the convergence characteristics of the improved MOL. The results for stress intensity factors (SIF) are also compared with the existing results obtained by finite element analysis.

The stress intensity factors are calculated for the compact tension specimen containing curved crack fronts. A thickness average stress intensity factor is evaluated and it is used to compare results for fracture toughness.

To explain the tunnelling behaviour in the compact tension specimen a complete elasto-plastic analysis is carried out, using the improved formulation. For a straight crack the load versus crack mouth opening displacement curve obtained by analysis is compared with the experimental plot. The experiments were conducted at NASA Lewis Research Center. The J-integral values at different thickness levels of the compact tension specimen are computed to predict the crack initiation through the thickness.

The results obtained show that the introduction of the five-point finite difference formulas has considerably enhanced the accuracy of the method of lines. A relatively coarse grid is sufficient to yield an accurate result.

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LIST OF SYMBOLS

a	crack length for the edge crack specimen
a_{av}	average crack length $\left(a_{av} = \frac{a_{25} + a_{50} + a_{75}}{3}\right)$
a_{25}	crack length measurement taken at midway between center and left surface of the three-dimensional edge crack specimen
a_{50}	crack length measurement taken at the center of the three-dimensional edge crack specimen
a_{75}	crack length measurement taken at midway between center and right surface of the three-dimensional edge crack specimen
a_{100}	crack length measurement taken at the right surface of the three-dimensional edge crack specimen
A_1, A_2, A_3	coefficient matrices for the first order ordinary differential equation
$A\ell$	Aluminum alloy
B	Thickness of the three-dimensional specimen
E	Young's modulus of elasticity
G	Shear modulus of elasticity
h_i	nodal spacing used in recurrence relation method
h_x, h_y, h_z	line spacings along Cartesian coordinate axes
$\bar{h}_x, \bar{h}_y, \bar{h}_z$	nondimensional line spacings along Cartesian coordinate axes
$[I]$	identity matrix
J	two-dimensional Rice integral
$\vec{J}$	general three-dimensional J-integral vector

J_x	x-directional component of J-integral vector $\vec{J}$
$\bar{J}$	thickness average J-integral value
J^*	the numerical value of J-integral at which crack initiation starts
K_I	stress intensity factor for opening mode
K_{av}	thickness average stress intensity factor
$K_{av}(SC)$	thickness average stress intensity factor for a straight crack front in a three-dimensional specimen
K_C	stress intensity factor at the center of a three-dimensional specimen
K_C (SC)	stress intensity factor at the center of a three-dimensional specimen, containing a crack with straight front
K_S	stress intensity factor at the surface of a three-dimensional specimen
$K_S(SC)$	stress intensity factor at the surface of a three dimensional specimen, containing a crack with straight front
δK	nondimensional difference between the thickness average stress intensity factors for a straight crack and a curved crack $\{\delta K = \dfrac{K_{av}(CCF) - K_{av}(SC)}{K_{av}(SC)}\}$
$K_{av}(CCF)$	thickness average stress intensity factor for a curved crack front
$[K_x],[K_y],[K_z]$	coefficient matrices of second order differential equations in Cartesian coordinates
$[\bar{K}_x],[\bar{K}_y],[\bar{K}_z]$	nondimensional coefficient matrices of second order differential equations in Cartesian coordinates

ℓ	number of lines in x direction
$\Delta\ell$	crack tunnel depth defined as the difference of two crack measurements, taken at the center and the surface of a three-dimensional specimen ($\Delta\ell = a_{50}-a_{100}$)
L	total length of the cracked specimen
m	number of lines in y direction
n	number of lines in z direction
n_j	the outward normal to a surface
NX,NY,NZ	number of lines in a given plane
NXC	number of lines on the cracked plane
P	total load on a specimen
r	crack edge position correction, measured from the originally assumed midpoint position
$\{r(x)\}$	coupling vector for x-directional second order differential equations
$\{\bar{r}(\bar{x})\}$	coupling vector for x-directional second order differential equations for elasto-plastic case
$\{\bar{r}_1^p(\bar{x})\}$	vector containing nondimensional plastic strain terms for x-directional second order differential equations
$\{\bar{r}_2^p(\bar{x})\}$	vector containing nondimensional shearing plastic strain terms for x-directional second order differential equations
R	distance from crack edge
R(x)	coupling vector for x-directional first order differential equations
$\bar{R}(\bar{x})$	coupling vector for x-directional first order differential equations for elasto-plastic case

$\bar{R}_1^p(\bar{x})$	vector containing nondimensional plastic strain terms for x-directional first order differential equations
$\bar{R}_2^p(\bar{x})$	vector containing nondimensional shearing plastic strain terms for x-directional first order differential equations
$\{s(y)\}$	coupling vector for y-directional second order differential equations
$\{\bar{s}(\bar{y})\}$	coupling vector for y-directional second order differential equations for elasto-plastic case
$\{\bar{s}_1^p(\bar{y})\}$	vector containing nondimensional plastic strain terms for y-directional second order differential equations
$\{\bar{s}_2^p(\bar{y})\}$	vector containing nondimensional shearing plastic strain terms for y-directional second order differential equations
$\{s^*(y)\}$	vector for y-directional second order differential equations for a compact tension specimen
$S(y)$	coupling vector for y-directional first order differential equations
$\bar{S}(\bar{y})$	coupling vector for y-directional first order differential equations for elasto-plastic case
$\bar{S}_1^p(\bar{y})$	vector containing nondimensional plastic strain terms for y-directional first order differential equations
$\bar{S}_2^p(\bar{y})$	vector containing nondimensional shearing plastic strain terms for y-directional first order differential equations
SC	abbreviation for straight crack
$\{t(z)\}$	coupling vector for z-directional second order differential equations
$\{\bar{t}(\bar{z})\}$	coupling vector for z-directional second order differential equations for elasto-plastic case

$\{\bar{t}_1^p(\bar{z})\}$	vector containing nondimensional plastic strain terms for z-directional second order differential equations
$\{\bar{t}_2^p(\bar{z})\}$	vector containing nondimensional shearing plastic strain terms for z-directional second order differential equations
$T(z)$	coupling vector for z-directional first order differential equations for elasto-plastic case
$\bar{T}(\bar{z})$	coupling vector for z-directional first order differential equations for elasto-plastic case
$\bar{T}_1^p(\bar{z})$	vector containing nondimensional plastic strain terms for z-directional first order differential equations
$\bar{T}_2^p(\bar{z})$	vector containing nondimensional shearing plastic strain terms
T_i	component of a traction vector
u	x-directional displacement
$\bar{u}$	nondimensional x-directional displacement
u_i	component of the displacement vector
$\bar{u}_i$	nondimensional component of the displacement vector
U	vector containing x-directional dependent variable in the first order differential equations
$\bar{U}$	vector containing nondimensional x-directional dependent variable in the first order differential equation for elasto-plastic case
v	y-directional displacement
$\bar{v}$	nondimensional y-directional displacement
V	vector containing y-directional dependent variable in the first order differential equations

$\bar{V}$	vector containing nondimensional y-directional dependent variable in the first order differential equation for elasto-plastic case
w	z-directional displacement
$\bar{w}$	nondimensional z-directional displacement
W	width of the three-dimensional cracked specimen
W'	vector containing z-directional dependent variable in the first order differential equation
$\bar{W}$	vector containing nondimensional z-directional dependent variable in the first order differential equation for elasto-plastic case
$W(\varepsilon)$	strain energy density function
x,y,z	rectangular Cartesian coordinates
$\bar{x},\bar{y},\bar{z}$	nondimensional rectangular Cartesian coordinates
δ_{ij}	Kronecker delta
λ	Lame's constant
ν	Poisson's ration
ρ	distance from the crack tip
η	curvature parameter $\left(\eta = \frac{a_{av} - a_{100}}{a_{av}}\right)$
σ	applied uniform surface stress
σ_t	tensile stress in the uniaxial stress-strain test
σ_o	tensile yield stress
σ_e	equivalent stress

σ_{ij}	components of stress tensor
$\sigma_x, \sigma_y, \sigma_z$ $\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$	components of stress tensor in Cartesian coordinates
$\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\sigma}_{zz}$ $\bar{\sigma}_{xy}, \bar{\sigma}_{yz}, \bar{\sigma}_{zx}$	components of nondimensional stress tensor in Cartesian coordinates
ε_t	strain in the uniaxial stress strain test
ε_{ij}	components of strain tensor
$\Delta\varepsilon_{ij}$	components of incremental strain tensor
$\bar{\varepsilon}_{ij}$	components of nondimensional strain tensor
$\bar{\varepsilon}_{xx}, \bar{\varepsilon}_{yy}, \bar{\varepsilon}_{zz}$, $\bar{\varepsilon}_{xy}, \bar{\varepsilon}_{yz}, \bar{\varepsilon}_{zx}$	components of nondimensional strain tensor in Cartesian coordinates
$\Delta\bar{\varepsilon}_{xx}, \Delta\bar{\varepsilon}_{yy}, \Delta\bar{\varepsilon}_{zz}$ $\Delta\bar{\varepsilon}_{xy}, \Delta\bar{\varepsilon}_{zx}, \Delta\bar{\varepsilon}_{yz}$	components of nondimensional incremental strain tensor in Cartesian coordinates
$\Delta\bar{\varepsilon}_p$	equivalent nondimensional plastic strain increment
$\bar{\varepsilon}'_{ij}$	modified nondimensional total strain tensor
$\bar{\varepsilon}'_x, \bar{\varepsilon}'_y, \bar{\varepsilon}'_z$, $\bar{\varepsilon}'_{xy}, \bar{\varepsilon}'_{yz}, \bar{\varepsilon}'_{zx}$	components of nondimensional total strain tensor in Cartesian coordinates
$\bar{\varepsilon}_{et}$	equivalent modified nondimensional total strain
$\tau(y)$	applied parabolically varying shear traction
τ_1	maximum value of applied shear traction

Subscripts:

x,y,z	refer to rectangular Cartesian coordinates
i,j,k	integer
I,II,III IV,V,VI	refer to plane numbering on the surface of the specimen

Superscripts:

–	nondimensional variable
•	derivative of displacements with respect to independent variable in corresponding directions
e	elastic
p	plastic

CHAPTER 1

INTRODUCTION

Knowledge of the stress and strain distributions in the neighborhood of a singularity such as the tip of a crack in a beam or compact tension specimen is of fundamental importance in evaluating the resistance to fracture of structural materials. In the past various researchers (1,2) using stress functions and complex variable approach, solved plane elastic problem for cracked bodies. Gross et. al. (3) used boundary collocation technique to solve different planar crack problems. The success of both analytical and numerical methods have placed present fracture mechanics on a firm foundation.

However, many fracture mechanics applications do not involve the solution of plane problems. For example, the stress analysis of corner cracks or surface cracks in practical structures is a three-dimensional problem and rational approach to design of such structures requires an accurate method of three-dimensional analyses. As described in Reference 4, at present, there is a wide divergence among the various approximate solutions to such three dimensional crack problems and in these circumstances structural design using fracture mechanics must proceed largely on the basis of experience and laboratory modeling tests.

A closely related problem occurs during the fracture toughness testing of standard compact tension specimens as shown in figure 1(a). Very rigid standard have been imposed by ASTM (5), to ensure that

valid fracture toughness values will be obtained in a given test. One of the requirements is the production of a fatigue crack. However, the fatigue crack will usually not grow uniformly across the specimen thickness, i.e. the initially straight crack front will become curved. The crack length will thus vary across the specimen thickness, and some average value must be used to calculate the fracture toughness. Reference (5) provides a standard measure such that if the crack front curvature as measured after the specimen is broken, is greater than this standard measure, the test is invalid.

Recently, however, suggestions have been made (6) that the above standard is too rigid and can be relaxed since the effect of the curved crack front may be less than originally anticipated. These suggestions, however, are based on rather weak evidence (7) and it is not yet clear whether the E-399 standard should be relaxed or not in this respect. What is needed is a clear definitive analysis of the effect of the crack front curvature on the stress intensity factor. One of the objectives of the present investigation is to perform this analysis.

Initial attempts to solve fully three-dimensional elastic problems were made by applying certain boundary correction factors to the existing two dimensional results. These approximations were solely based on intuitive judgement and resulted in widely scattered solutions for the same problems. Three-dimensional problems are inherently more complex and in almost all the cases one has to resort to some sort of numerical technique which leads to a large

set of equations to be solved on a digital computer. Many numerical schemes are devised to obtain the stress and strain distribution in the neighborhood of a crack in a three-dimensional body. Notably among them are Boundary Element Method (8) and Finite Element Method (9,10,11). Significant progress has been reported in the development of both of these techniques. Special crack tip elements are developed which include the effect of crack tip singularity. However, the effect of element size and arrangement around the crack tip, on the convergence of the solution needs more extensive study.

A few attempts were made to determine stress fields near crack tips having some curvature by using experimental techniques. McGowan (12) used a 3-D stress freezing technique to investigate the effect of crack front curvature on the stress intensity factor distribution in a single edge notch specimen. He reported that the increase in the crack front curvature decreases the stress intensity factor (SIF) at the center of the specimen and also the thickness average stress intensity factor. In his study the value of SIF at the surface was insensitive to the change in crack front curvature as well as the Poisson's ratio. This study predicts the behavior, which is in good agreement with other analyses, but the actual results have discrepancy of about 35% with an existing finite element analysis. The discrepancy in the results has been attributed to the difference in the crack length in the two analyses.

Fourney (13) devised a new technique of analyzing three-dimensional problems utilizing scattered light speckle interferometry. He

also found that the stress intensity factor increases at the surface due to the presence of crack front curvature. The crack tip singularity was found to be strongest at the center. This was cited as the reason for tunnelling behavior in a compact tension specimen.

Peirera et. al. (14) carried out a finite element stress analysis of a compact tension specimen. They distorted the crack tip elements to model a circular type of crack front. It was found that the local values of SIF decreases at the center and it rises at the surface of a compact tension specimen. An average value of SIF based on energy considerations was also calculated. This value was found to be decreasing with increasing crack tunnel depth.

Neale (15) performed another finite element analysis for a non standard compact tension specimen (W/B = 8) containing a thumbnail crack. He showed that if the average crack length is calculated based on ASTM method, then the fracture toughness is overestimated. Both the previous analytical studies did not use the standard compact tension specimen as shown in figure 1(a). Peirera et. al. used a tensile loading which does not model the actual loading condition. In their studies a crack length to width ratio (a/W) of 0.25 was used, while ASTM standard prescribes a range of 0.45 to 0.55. In Neale's analysis the thickness of the specimen used was ¼ of the standard thickness specified in the ASTM standard. Consequently, the results of these analyses can not be directly applied to a standard compact tension specimen.

All these analyses predict that crack growth will initiate from the surface of the specimen due to the presence of maximum SIF there.

This finding is in direct contradiction with the experimental results, in which a tunnelling behavior is observed. Neale (16) tried to explain this anamoly on the basis of plasticity effects. He performed an approximate elasto-plastic analysis to improve the numerical values of SIF. His elastic-plastic thumbnail model indicates that the maximum stress intensity factor occurs in the central portion of the compact tension specimen which is in agreement with experimental observations. However, due to the approximate nature of the analysis, there is a need to perform an accurate elasto-plastic stress analysis to resolve this discrepancy. This is another major objective of the present investigation.

The recently developed method known in the literature as the Method of Lines (MOL) was selected to perform the elastic and elasto-plastic stress analysis of two specimens namely the edge notch specimen shown in figure 2 and the standard compact tension of figure 1(a). In this method there is no prior assumption on the stress field around the crack tip. There is an extensive literature, primarily of Russian origin for the method of lines, which has been summarized to 1965 by Liskovetes (17). The method has been used extensively in solving problems in the area of fluid mechanics.

Jones, South and Klunker (18) presented an analysis of convergence and stability for the case of linear elliptic partial differential equations solved by the MOL. They indicated that the results obtained are of sufficiently high accuracy. It was shown that if the region of interest is divided into sufficiently few strips by

the dividing lines then accurate solutions can be obtained by using higher order finite difference approximation. In the study by Kurtz et. al. (19) it was observed that in problems where standard techniques failed to converge the line method was able to produce results. Hopkins and Wait (20) compared the execution time of MOL with Galerkin and collocation techniques, which were viewed as finite element discretization. With very few exceptions, they found that the line method yields faster results as compared to the other two techniques for a set of coupled and uncoupled parabolic partial differential equations.

The line method lies midway between completely analytical methods and completely discrete methods such as finite differences. The basis of this technique is the substitution of finite differences for the derivatives with respect to all the independent variables except one for which the derivatives are retained. This approach replaces a given partial differential equation with a system of simultaneous ordinary differential equations whose solution can then be obtained by standard means. These equations describe the dependent variable along lines which are parallel to the coordinate in whose direction the derivatives were retained.

An inherent advantage of the line method over other numerical methods is that good results are obtained from the use of a relatively coarse grid. This use of a coarse grid is permissible because parts of the solutions may be obtained in terms of continuous functions. Additional accuracy in normal stress distributions is derived

from the fact that they are expressed as first-order derivatives of the displacements and these derivatives can be analytically evaluated. Inherently inaccurate numerical differentation is required only for evaluating the shear stresses, but this presents no important loss of accuracy in this study since they are usually an order of magnitude smaller than the normal stresses. For problems with geometric singularities, additional accuracy is derived from using a displacement formulation since the resulting deformations are not singular.

The method has been shown to be well suited to the solution of certain three-dimensional crack problems, as has been successfully demonstrated (21,22). Fu and Malik (23,24) are the first to report the application of MOL to elasto-plasticity. The formulation was presented in terms of displacements and their normal derivatives. These equations were solved by a combination of power series and modal matrix method.

Although the MOL has given very good results in a number of specific geometries as described in the previous references, its use has been limited heretofore to bodies with rectangular boundaries (including the crack boundaries), or to the problems of axial symmetry. Thus, the very important surface flaw problem or curved crack front problem have not been treated. This was primarily due to the limitation in the number of lines that could be used, since the rate of convergence of the infinite series solutions of the differential equations decreases as the number of lines increased and as the dimensions of the specimen increased. It has also been shown in

reference (18) that the use of the MOL for solving elliptic partial differential equations can lead to problems of instability if the line spacing is made too small.

A third major objective of this investigation was therefore to reformulate the method to avoid or minimize the above difficulties. This was done by employing 5- point finite difference formulas rather than the usual 3- point formulas and by devising an algorithm for solving the three sets of ordinary differential equations for the two point boundary value problem using recurrence relations. These innovations have appreciably increased the applicability and accuracy of the MOL. The elastic formulation is then modified by including the plastic strain terms to solve an elasto-plastic problem.

A complete formulation in a rectangular cartesian coordinate system is presented for the newly improved MOL. The governing field equations are augmented to include the plastic strain terms. To study the convergence characteristics of the method, complete elastic solutions are obtained for different numbers of lines for an edge notch specimen. The stress intensity factors are computed for an edge notch specimen and a standard compact tension specimen. They are compared with the existing finite element analyses results to assess the validity of the MOL. The complete field solutions are evaluated for both the specimens for the different cases of crack front curvature. For compact tension specimen of a given material (Aℓ-5083) complete elastic-plastic stress analyses are carried out. Similar analyses are also done for specimens with different curved

crack fronts. Various fracture mechanics parameter such as the J- integral are evaluated to establish the effect of crack front curvature.

CHAPTER 2

SOLUTION OF CRACK PROBLEM BY THE METHOD OF LINES

In this chapter a method of solution for three-dimensional elasto-static problems is described. All the underlying assumptions and the governing field equations are described. An improved method of lines, in which five point finite difference equations are used, is employed to obtain a set of second order ordinary differential equations from the governing coupled partial differential equations. The second order differential equations are rearranged to formulate a set of first order differential equations and these equations are then solved by a new algorithm involving recurrence relations, which was found to be a simpler and an efficient technique to solve two point boundary value problems. The last section of the chapter deals with the evaluation of boundary vectors used in the recurrence relations for an edge notch specimen. The details are also given for the loading idealization for a compact tension specimen and its incorporation into the equations of the method of lines.

For all the elasto-static problems discussed in this thesis the following assumptions shall apply.

a) The deformations are infinitesimal.

b) All deformations are elastic.

c) Materials are isotropic and homogeneous.

d) Body forces are neglected.

2.1 Governing Equations

Within the framework of linearized elasticity theory, the field equations neglecting body forces are listed below.

The equilibrium equations are,

$$\sigma_{ij,j} = 0 \qquad i,j = 1,2,3 \tag{2.1}$$

where the standard tensor subscript notation is used. Hooke's law is,

$$\sigma_{ij} = \lambda\delta_{ij}\,\varepsilon_{kk} + 2\,G\,\varepsilon_{ij} \tag{2.2}$$

where λ and G are Lame's elastic constants. The strain-displacement relations are,

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3}$$

The solution must satisfy these equations at all interior points of the body and, in addition the prescribed conditions must be met on the boundaries. The above three sets of equations are combined to form three partial differential equations in terms of displacements. The resulting equations which are known as the Navier equations of equilibrium are,

$$G\,u_{i,jj} + (\lambda + G)\,u_{j,ji} = 0 \quad i,j = 1,2,3, \tag{2.4}$$

For problems formulated in rectangular cartesian coordinates, the equations can be written as,

$$\frac{\partial^2 u}{\partial x^2} + C_1 \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = C_2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{2.5}$$

$$\frac{\partial^2 v}{\partial y^2} + C_1 \left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial x^2}\right) = C_2 \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \tag{2.6}$$

$$\frac{\partial^2 w}{\partial z^2} + C_1 \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right) = C_2 \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{2.7}$$

where $$C_1 = \frac{1 - 2\nu}{2(1 - \nu)} \quad \text{and} \quad C_2 = - \frac{1}{2(1 - \nu)} \tag{2.8}$$

ν being Poisson's ratio.

The stress-displacement relations in cartesian coordinates are,

$$\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[(1 - \nu) \frac{\partial u}{\partial x} + \nu \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] \tag{2.9}$$

$$\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[(1 - \nu) \frac{\partial v}{\partial y} + \nu \left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\right] \tag{2.10}$$

$$\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[(1 - \nu) \frac{\partial w}{\partial z} + \nu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \tag{2.11}$$

$$\sigma_{xy} = \frac{E}{2(1 + \nu)} \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \tag{2.12}$$

$$\sigma_{yz} = \frac{E}{2(1 + \nu)} \left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] \tag{2.13}$$

$$\sigma_{zx} = \frac{E}{2(1 + \nu)} \left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] \tag{2.14}$$

The method of lines is used to solve the three coupled partial differential equations given by equations (2.5) through (2.7). For each problem appropriate boundary conditions are satisfied. In the next section the line method is described in detail.

2.2 Method of Lines

The recently developed Method of Lines (MOL) has been shown to be well suited to the solution of certain three-dimensional crack problems. In this method there is no prior assumption on the stress field around the crack tip. The line method lies midway between completely analytical and grid methods. The basis of the method is substitution of finite differences for the derivatives with respect to all the independant variables except one, with respect to which the derivatives are retained. This approach replaces a given partial differential equation by a system of simultaneous ordinary differential equation whose solution can then be obtained by standard means. The equations describe the dependent variable along lines which are parallel to the coordinate in whose direction the derivatives were retained.

Since in three-dimensional elasticity problems solutions of three partial differential equations are desired, three sets of parallel lines are constructed. An arbitrary grid consisting of these three sets of parallel lines is shown in figure 3. The lines parallel to the x axis are numbered as 1,2,3, -- NY, NY + 1 -- 2NY, 2NY + 1 -- 3NY, 3NY + 1 -- ℓ. The lines parallel to the y axis are numbered 1,2,3, -- NZ, NZ + 1 -- 2NZ, 2NZ + 1 -- 3NZ, 3NZ + 1 -- m. Finally lines parallel to the z axis are numbered as 1,2,3, -- NX,

NX + 1 -- 2NX, 2NX + 1 -- 3NX, 3NX + 1 -- n. This numbering system is chosen so that the resulting variables in the computer listings are identified through double subscripts only. The first subscript identifies the line along which the variables are calculated while the second subscript indicates the position along the line. In this work the lines are constantly spaced with spacings h_x, h_y and h_z. This is done purely for convenience. The advantage of uniform line spacing is that the resulting ordinary differential equations can be solved more easily than those that are derived fron non-uniform line spacing.

The equations of 3-D elasticity are the three coupled partial differential equations (2.5-2.7). In this case solutions for the dependent variables are possible only at points where the three sets of parallel lines intersect. These points are usually called nodes in a discretized body. This limitation is the result of coupling among the equations, which makes the particular selection of ordinary differential equations valid only at the nodes. The equations are developed for a crack specimer as shown in Figure 4. Since we have a three fold symmetry, only one quarter of the specimen is considered. One should also note the numbering of the various planes.

2.2.1 Reduction of the First Navier-Cauchy Equation and Associated Boundary Conditions for the Cracked Specimen

For the solution of equation (2.5), the lines parallel to x axis in Figure 3 are considered. The x directional displacements of points along these lines will be denoted as u_1, u_2, -- u_ℓ. We define $\dot{v}|_1$, $\dot{v}|_2$, $\dot{v}|_3$ . . . $\dot{v}|_\ell$ as the derivatives of the y directional displacements of the same points on these lines with respect to y and $\dot{w}|_1$,

$\dot{w}|_2, \dot{w}|_3 \ldots \dot{w}|_\ell$ as the derivatives of the z directional displacements of the same points on these lines with respect to z. These displacements and derivatives can then be regarded as a function of x only since they are variables upon lines which are parallel to the x axis.

By the above given definiton, the ordinary differential equation along a generic line ij in figure 5 may be written as,

$$\frac{d^2u_{ij}}{dx^2} + C_1 \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)_{ij} = C_2 \frac{d}{dx} \dot{v}|_{ij} + C_2 \frac{d}{dx} \dot{w}|_{ij} \tag{2.15}$$

Introducing 5- point finite differences, the partial derivatives of u with respect to y and z along the x directional line (ij) of figure 5 can be written as follows,

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{ij} \simeq \frac{1}{12h_y^2} \left(-u_{i-2,j} + 16u_{i-1,j} - 30u_{ij} + 16u_{i+1,j} - u_{i+2,j}\right) \tag{2.16}$$

and

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_{ij} \simeq \frac{1}{12h_z^2} \left(-u_{i,j-2} + 16u_{i,j-1} - 30u_{ij} + 16u_{i,j+1} - u_{i,j+2}\right) \tag{2.17}$$

On substituting equations (2.16) and (2.17) into equation (2.15), the general equation along interior lines is obtained. Thus,

$$\frac{d^2u_{ij}}{dx^2} + C_1\left[\frac{1}{12h_y^2} \left(-u_{i-2,j} + 16u_{i-1,j} - 30u_{ij} + 16u_{i+1,j} - u_{i+2,j}\right) + \frac{1}{12h_z^2} \times \left(-u_{i,j-2} + 16u_{i,j-1} - 30u_{ij} + 16u_{i,j+1} - u_{i,j+2}\right)\right] = C_2 \frac{d}{dx} (\dot{v} + \dot{w})_{ij} \tag{2.18}$$

The rearrangement of the above equation leads to,

$$\frac{d^2u_{ij}}{dx^2} = \frac{C_1}{12h_y^2} u_{i-2,j} - \frac{4C_1}{3h_y^2} u_{i-1,j} + (\frac{5C_1}{2h_y^2} + \frac{5C_1}{2h_z^2}) u_{ij} - \frac{4C_1}{3h_y^2} X$$

$$u_{i+1,j} + \frac{C_1}{12h_y^2} u_{i+2,j} + \frac{C_1}{12h_z^2} u_{i,j-2} - \frac{4C_1}{3h_z^2} u_{i,j-1} - \frac{4C_1}{3h_z^2} u_{i,j+1}$$

$$+ \frac{C_1}{12h_z^2} u_{i,j+2} + C_2 \frac{d}{dx}(\dot{v} + \dot{w})_{ij} \tag{2.19}$$

Similar differential equations are obtained for the other displacements of the points on the x directional lines. Since each equation has the terms of the displacements of the points on the surrounding lines, these equations constitute a system of ordinary differential equations for the displacements $u_1, u_2 \ldots u_\ell$.

The equation (2.19) is applicable only to interior lines. For boundary surface lines and lines adjacent to the boundary surface lines, the difference expressions for the second derivative will involve imaginary lines outside the boundary. As shown in figure 6, the imaginary lines below the plane V are designated as 1^{fy}, $NY+1^{fy}$ . . . $(\ell - NY+1)^{fy}$. The imaginary lines lying on the planes adjacent to plane VI are numbered as 1^{f2z} . . . NY^{f2z} and 1^{f1z} . . . NY^{f1z} respectively. Similar numbering procudure is followed for the imaginary lines lying on the planes, which are located adjacent to planes II and III respectively. The arrangement of the planes V, VI, II and III is shown in figure 4. Since three-dimensional elasticity problems have three boundary conditions at every point of the boundary surface and a

second order ordinary differential equation needs only two conditions, the shear stresses at the boundaries are used to eliminate the imaginary lines outside the surface. While the condition of the prescribed normal traction or displacement will be enforced through the constants of the homogeneous solutions. The method used to derive the equations for boundary surface lines and line adjacent to it, is described in detail in the Appendix A. Appropriate difference equations are used to express the partial derivatives. For example, to write the difference equations for the partial derivatives with respect to z for the lines ℓ - NY+1 through ℓ, a backward finite difference formula is used due to the lack of sufficient grid points to write central difference equations. On following the procedure described in Appendix A, the differential equation for line 1 can be written as,

$$\frac{d^2u_1}{dx^2} + \frac{C_1}{12h_y^2}[-20u_1 + 17u_2 + 4u_3 - u_4]$$

$$+ \frac{C_1}{12h_z^2}[-30u_1 + 30u_{NY+1} - 2u_{2NY+1}]$$

$$= C_2 \frac{d}{dx}(\dot{u} + \dot{w})_1 - \frac{11C_1}{6h_y}\frac{dv}{dx}\Big|_1 \qquad (2.20)$$

similarly the differential equation for the line ℓ - 2NY+1 is

$$\frac{d^2u_{\ell-2NY+1}}{dx^2} + \frac{C_1}{12h_y^2}[-20u_{\ell-2NY+1} + 17u_{\ell-2NY+2}$$

$$+ 4u_{\ell-2NY+3} - u_{\ell-2NY+4}] + \frac{C_1}{12h_z^2} X$$

$$[-u_{\ell\,-\,4NY+1} + 16u_{\ell\,-\,3NY+1} - 31u_{\ell\,-\,2NY+1} + 16u_{\ell\,-\,NY-1}]$$

$$= C_2 \frac{d}{dx}(\dot{v} + \dot{w})_{\ell\,-\,2NY+1} - \frac{C_1}{6h_y}\frac{dv}{dx}\Big|_{\ell\,-\,2NY+1} - \frac{C_1}{6h_z}\frac{dw}{dx}\Big|_{\ell\,-\,NY+1} \tag{2.21}$$

Finally, introducing matrix notation all the ordinary differential equations along the x directional lines are expressed in the form,

$$\underset{\ell \times 1}{\frac{d^2\{u\}}{dx^2}} = \underset{\ell \times \ell}{[K_x]}\ \underset{\ell \times 1}{\{u\}} + \frac{d}{dx}\ \underset{\ell \times 1}{\{r(x)\}} \tag{2.22}$$

where the matrix $[K_x]$ and the column vectors $\{u\}$ and $\{r(x)\}$ are given below.

$$\underset{\ell \times \ell}{[K_x]} = \left[\begin{array}{c|c|c|c|c|c|c}
\underset{NY \times NY}{[K_{11}]} & \underset{NY \times NY}{[K_{12}]} & \underset{NY \times NY}{[K_{13}]} & 0 & 0 & 0 & 0 \\ \hline
\underset{NY \times NY}{[K_{21}]} & \underset{NY \times NY}{[K_{22}]} & \underset{NY \times NY}{[K_{21}]} & \underset{NY \times NY}{[K_{24}]} & 0 & 0 & 0 \\ \hline
\underset{NY \times NY}{[K_{24}]} & \underset{NY \times NY}{[K_{21}]} & \underset{NY \times NY}{[K_{11}]} & \underset{NY \times NY}{[K_{21}]} & \underset{NY \times NY}{[K_{24}]} & 0 & 0 \\ \hline
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \hline
0 & 0 & \underset{NY \times NY}{[k_{24}]} & \underset{NY \times NY}{[K_{21}]} & \underset{NY \times NY}{[K_{11}]} & \underset{NY \times NY}{[K_{21}]} & \underset{NY \times NY}{[K_{24}]} \\ \hline
0 & 0 & 0 & \underset{NY \times NY}{[K_{24}]} & \underset{NY \times NY}{[K_{21}]} & \underset{NY \times NY}{[K_{22}]} & \underset{NY \times NY}{[K_{21}]} \\ \hline
0 & 0 & 0 & \underset{NY \times NY}{[K_{24}]} & \underset{NY \times NY}{[K_{NZ,NZ-2}]} & \underset{NY \times NY}{[K_{NZ,NZ-1}]} & \underset{NY \times NY}{[K_{NZ,NZ}]}
\end{array}\right] \tag{2.23}$$

Where the submatrices are given as follows,

$$
\underset{NY\times NY}{[K_{11}]} =
\begin{bmatrix}
\frac{5C_1}{3h_y^2}+\frac{5C_1}{2h_z^2} & -\frac{17C_1}{12h_y^2} & -\frac{C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 & 0 \\
-\frac{4C_1}{3h_y^2} & \frac{31C_1}{12h_y^2}+\frac{5C_1}{2h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{5C_1}{2h_y^2}+\frac{5C_1}{2h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{5C_1}{2h_y^2}+\frac{5C_1}{2h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} \\
0 & 0 & 0 & \frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{31C_1}{12h_y^2}+\frac{5C_1}{2h_z^2} & -\frac{4C_1}{3h_y^2} \\
0 & 0 & 0 & \frac{C_1}{12h_y^2} & -\frac{C_1}{3h_y^2} & -\frac{17C_1}{12h_y^2} & \frac{5C_1}{3h_y^2}+\frac{5C_1}{2h_z^2}
\end{bmatrix}
$$

$$[K_{12}] = -\frac{8}{3}\frac{C_1}{h^2_z}[I]$$

$$[K_{13}] = \frac{C_1}{6h^2_z}[I]$$

$$[K_{21}] = -\frac{C_1}{3h^2_z}[I]$$

$$[K_{24}] = \frac{C_1}{12h^2_z}[I]$$

$$[K_{NZ,NZ-2}] = -\frac{C_1}{3h^2_z}[I]$$

$$[K_{NZ,NZ-1}] = -\frac{17C_1}{12h^2_z}[I]$$

Matrix [I] is an indentity matrix of order NY x NY

Note that C_1 is $\frac{1-2\nu}{2(1-\nu)}$

$$
\underset{NY \times NY}{[K_{22}]} = \begin{bmatrix}
\frac{5C_1}{3h_y^2} + \frac{31C_1}{12h_z^2} & -\frac{17C_1}{12h_y^2} & -\frac{C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 & 0 \\
-\frac{4C_1}{3h_y^2} & \frac{31C_1}{12h_y^2} + \frac{31C_1}{12h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{5C_1}{2h_y^2} + \frac{31C_1}{12h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{5C_1}{2h_y^2} + \frac{31C_1}{12h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} \\
0 & 0 & 0 & \frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{31C_1}{12h_y^2} + \frac{31C_1}{12h_z^2} & -\frac{4C_1}{3h_y^2} \\
0 & 0 & 0 & \frac{C_1}{12h_y^2} & -\frac{C_1}{3h_y^2} & -\frac{17C_1}{12h_y^2} & \frac{5C_1}{3h_y^2} + \frac{31C_1}{12h_z^2}
\end{bmatrix}
$$

$$
[K_{NZ,NZ}]_{NY\times NY} = \begin{bmatrix}
\frac{5C_1}{3h_y^2}+\frac{5C_1}{3h_z^2} & -\frac{17C_1}{12h_y^2} & -\frac{C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 & 0 \\
-\frac{4C_1}{3h_y^2} & \frac{31C_1}{12h_y^2}+\frac{5C_1}{3h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{5C_1}{2h_y^2}+\frac{5C_1}{3h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{5C_1}{2h_y^2}+\frac{5C_1}{3h_z^2} & -\frac{4C_1}{3h_y^2} & \frac{C_1}{12h_y^2} \\
0 & 0 & 0 & \frac{C_1}{12h_y^2} & -\frac{4C_1}{3h_y^2} & \frac{31C_1}{12h_y^2}+\frac{5C_1}{3h_z^2} & -\frac{4C_1}{3h_y^2} \\
0 & 0 & 0 & \frac{C_1}{12h_y^2} & -\frac{C_1}{3h_y^2} & -\frac{17C_1}{12h_y^2} & \frac{5C_1}{3h_y^2}+\frac{5C_1}{3h_z^2}
\end{bmatrix}
$$

$$\underset{\ell x1}{\{u\}} = \begin{bmatrix} \{u\}_1 \\ \{u\}_2 \\ \vdots \\ \{u\}_{NZ-1} \\ \{u\}_{NZ} \end{bmatrix} \qquad \underset{\ell x1}{\{r(x)\}} = \begin{bmatrix} \{r_1\ (x)\} \\ \{r_2\ (x)\} \\ \vdots \\ \{r_{NZ-1}(x)\} \\ \{r_{NZ}\ (x)\} \end{bmatrix} \tag{2.24}$$

where the partitioned column vectors are

$$\underset{NYx1}{\{u\}_1} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{NY-1} \\ u_{NY} \end{bmatrix} \qquad \underset{NYx1}{\{u\}_2} = \begin{bmatrix} u_{NY+1} \\ u_{NY+2} \\ \vdots \\ u_{2NY-1} \\ u_{2NY} \end{bmatrix}$$

$$
\underset{NY\times 1}{\{u\}_{NZ-1}} = \begin{bmatrix} u_{\ell-2NY+1} \\ u_{\ell-2NY+2} \\ \vdots \\ u_{\ell-NY-1} \\ u_{\ell-NY} \end{bmatrix} \qquad \underset{NY\times 1}{\{u\}_{NZ}} = \begin{bmatrix} u_{\ell-NY+1} \\ u_{\ell-NY+2} \\ \vdots \\ u_{\ell-1} \\ u_{\ell} \end{bmatrix}
$$

$$
\underset{NY\times 1}{\{r_1(x)\}} = \begin{bmatrix} C_2\ (\dot{v} + \dot{w})_1 - \left.\frac{11C_1 v}{6h_y}\right|_1 \\ C_2\ (\dot{v} + \dot{w})_2 + \left.\frac{C_1 v}{6h_y}\right|_1 \\ C_2\ (\dot{v} + \dot{w})_3 \\ \vdots \\ C_2\ (\dot{v} + \dot{w})_{NY-2} \\ C_2\ (\dot{v} + \dot{w})_{NY-1} - \left.\frac{C_1 v}{6h_y}\right|_{NY} \\ C_2\ (\dot{v} + \dot{w})_{NY} + \left.\frac{11C_1 v}{6h_y}\right|_{NY} \end{bmatrix}
$$

$$\underset{NYx1}{\{r_2(x)\}} = \begin{bmatrix} C_2\ (\dot{v} + \dot{w})_{NY+1} - \left.\dfrac{11C_1 v}{6h_y}\right|_{NY+1} \\[2ex] C_2\ (\dot{v} + \dot{w})_{NY+2} + \left.\dfrac{C_1 v}{6h_y}\right|_{NY+1} \\[2ex] C_2(\dot{v} + \dot{w})_{NY+3} \\ \vdots \\ C_2\ (\dot{v} + \dot{w})_{2NY-2} \\[2ex] C_2\ (\dot{v} + \dot{w})\ _{NY-1} - \left.\dfrac{C_1 v}{6h_y}\right|_{2NY} \\[2ex] C_2\ (\dot{v} + \dot{w})_{2NY} + \left.\dfrac{11C_1 v}{6h_y}\right|_{2NY} \end{bmatrix}$$

$$
\{r_{NZ-1}(x)\}_{NY\times 1} = \begin{bmatrix}
C_2\,(\dot{v}+\dot{w})_{\ell-2NY+1} - \frac{11C_1 v}{6h_y}\Big|_{\ell-2NY+1} - \frac{C_1 w}{6h_z}\Big|_{\ell-NY+1} \\
C_2\,(\dot{v}+\dot{w})_{\ell-2NY+2} + \frac{C_1 v}{6h_y}\Big|_{\ell-2NY+1} - \frac{C_1 w}{6h_z}\Big|_{\ell-NY+2} \\
C_2\,(\dot{v}+\dot{w})_{\ell-2NY+3} - \frac{C_1 w}{6h_z}\Big|_{\ell-NY+3} \\
\vdots \\
C_2\,(\dot{v}+\dot{w})_{\ell-NY-2} - \frac{C_1 w}{6h_z}\Big|_{\ell-2} \\
C_2(\dot{v}+\dot{w})_{\ell-NY-1} - \frac{C_1 v}{6h_y}\Big|_{\ell-NY} - \frac{C_1 w}{6h_z}\Big|_{\ell-1} \\
C_2\,(\dot{v}+\dot{w})_{\ell-NY} + \frac{11C_1 v}{6h_y}\Big|_{\ell-NY} - \frac{C_1 w}{6h_z}\Big|_{\ell}
\end{bmatrix}
$$

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$$\underset{NY\times 1}{\{r_{NZ}(x)\}} = \begin{bmatrix} C_2\,(\dot{v}+\dot{w})_{\ell-NY+1} - \left.\dfrac{11C_1 v}{6h_y}\right|_{\ell-NY+1} + \left.\dfrac{11C_1 w}{6h_z}\right|_{\ell-NY+1} \\[2ex] C_2\,(\dot{v}+\dot{w})_{\ell-NY+2} + \left.\dfrac{C_1 v}{6h_y}\right|_{\ell-NY+1} + \left.\dfrac{11C_1 w}{6h_z}\right|_{\ell-NY+2} \\[2ex] C_2\,(\dot{v}+\dot{w})_{\ell-NY+3} + \left.\dfrac{11C_1 w}{6h_z}\right|_{\ell-NY+3} \\ \vdots \\ C_2\,(\dot{v}+\dot{w})_{\ell-2} + \left.\dfrac{11C_1 w}{6h_z}\right|_{\ell-2} \\[2ex] C_2\,(\dot{v}+\dot{w})_{\ell-1} - \left.\dfrac{C_1 v}{6h_y}\right|_{\ell} + \left.\dfrac{11C_1 w}{6h_z}\right|_{\ell-1} \\[2ex] C_2\,(\dot{v}+\dot{w})_{\ell} + \left.\dfrac{11C_1 v}{6h_y}\right|_{\ell} + \left.\dfrac{11C_1 w}{6h_z}\right|_{\ell} \end{bmatrix}$$

2.2.2 Reduction of the Second Navier-Cauchy Equation and Associated Boundary Conditions for the Cracked Specimen

For the solution of equation 2.6, the lines parallel to y- axis in figure 3 are considered. The y directional displacements of points along these lines will be denoted as $v_1, v_2, v_3, \ldots, v_m$. We define $\dot{u}|_1, \dot{u}|_2, \dot{u}|_3, \ldots, \dot{u}|_m$ as the derivatives of the x directional displacements of the same points on these lines with respect to x and $\dot{w}|_1, \dot{w}|_2, \dot{w}|_3, \ldots, \dot{w}|_m$ as the derivatives of the z directional displacements of the same points on these lines with respect to z. These displacements and derivatives can then be regarded as functions of y only since they are variables upon lines which are parallel to the y axis.

By the above given definition, the ordinary differential equation along a generic line ij may be written as,

$$\frac{d^2 v_{ij}}{dy^2} + C_1 \left(\frac{d^2 v}{dx^2} + \frac{d^2 v}{dz^2}\right)_{ij} = C_2 \frac{d}{dy} \dot{u}|_{ij} + C_2 \frac{d}{dy} \dot{w}|_{ij} \tag{2.25}$$

Introducing 5- point finite differences, the partial derivatives of v along the y directional line ij can be written as follows,

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{ij} \simeq \frac{1}{12h_x^2} (- v_{i-2,j} + 16v_{i-1,j} - 30v_{ij} + 16v_{i+1,j} - v_{i+2,j}) \tag{2.26}$$

and

$$\left(\frac{\partial^2 v}{\partial z^2}\right)_{ij} \simeq \frac{1}{12h_z^2} (- v_{i,j-2} + 16v_{i,j-1} - 30v_{ij} + 16v_{i,j+1} - v_{i,j+2}) \tag{2.27}$$

On substituting equations (2.26) and (2.27) into equation (2.25) the general equation along interior lines is obtained. Thus,

$$\frac{d^2 v_{ij}}{dy^2} + C_1 \Big[\frac{1}{12h_x^2}(-v_{i-2,j} + 16v_{i-1,j} - 30v_{ij} + 16v_{i+1,j} - v_{i+2,j})$$

$$+ \frac{1}{12h_z^2}(-v_{i,j-2} + 16v_{i,j-1} - 30v_{ij} + 16v_{i,j+1} - v_{i,j+2})\Big]$$

$$= C_2 \frac{d}{dy}(\dot{u} + \dot{w})_{ij} \tag{2.28}$$

The rearrangement of the above equation leads to

$$\frac{d^2 v_{ij}}{dy^2} = \frac{C_1}{12h_x^2} v_{i-2,j} - \frac{4C_1}{3h_x^2} v_{i-1,j} + \left(\frac{5C_1}{2h_x^2} + \frac{5C_1}{2h_z^2}\right) v_{ij}$$

$$- \frac{4C_1}{3h_z^2} v_{i,j+1} + \frac{C_1}{12h_z^2} v_{i,j+2} + \frac{C_1}{12h_z^2} v_{i,j-2} - \frac{4C_1}{3h_z^2} v_{i,j-1}$$

$$+ C_2 \frac{d}{dy}(\dot{u} + \dot{w})_{ij} - \frac{4C_1}{3h_x^2} v_{i+1,j} + \frac{C_1}{12h_x^2} v_{i+2,j} \tag{2.29}$$

Similar differential equations are obtained for the other displacements of the points on the y directional lines. Since each equation has the terms of the displacemnts of the points on the surrounding lines, these equations constitute a system of ordinary differential equations for the displacements $v_1, v_2 \ldots v_m$.

The equation (2.29) is applicable only to interior lines. For the boundary surface lines and lines adjacent to the boundary surface lines, the finite difference expressions for the second derivative will involve the displacements on imaginary lines lying outside the boundary as shown in figure 7. The shear stress at the boundaries is used to eliminate the displacements on these imaginary lines near the surface. While the condition of prescribed normal traction or displacement wil be enforced through the constants of the homogeneous solutions.

A detailed description of the method used in deriving the equations for the lines near the boundary, can be found in the Appendix A. The use of fictitious lines and their subsequent elimination using the conditions of symmetry and prescribed shear stress at the surface leads to the following equations for line 1,

$$\frac{d^2 v_1}{dy^2} + \frac{C_1}{12h_x^2} [-20v_1 + 17v_{NZ+1} + 4v_{2NZ+1} - v_{3NZ+1}]$$

$$+ \frac{C_1}{12h_z^2} [-30v_1 + 32v_2 - 2v_3]$$

$$= C_2 \frac{d}{dy} (\dot{u} + \dot{w})_1 - \frac{11C_1}{6h_x} \Big|_1 \qquad (2.30)$$

on following the same procedure the ordinary differential equation for line m can be written as,

$$\frac{d^2 v_m}{dy^2} + \frac{C_1}{12h_x^2} [-v_{m-3NZ} + 4v_{m-2NZ} + 17v_{m-NZ} - 20v_m]$$

$$+ \frac{C_1}{12h_z^2} [-v_{m-3} + 4v_{m-2} + 17v_{m-1} - 20v_m]$$

$$= C_2 \frac{d}{dy} (\dot{u} + \dot{w})_m + \frac{11C_1}{6h_x} \frac{du}{dy} \Big|_m + \frac{11C_1}{6h_z} \frac{dw}{dy} \Big|_m \qquad (2.31)$$

The comparison of equations (2.30) and (2.31) with equation (2.28) shows the presence of additional terms on the right-hand side. These terms appear here due to the consideration of imaginary lines outside the boundary surface.

Finally, introducing matrix notation, all the ordinary differential equations along the y- directional lines are expressed in the form

$$\underset{m\times 1}{\frac{d^2\{v\}}{dy^2}} = \underset{m\times m}{[K_y]} \ \underset{m\times i}{\{v\}} + \frac{d}{dy} \underset{m\times 1}{\{s(y)\}} \qquad (2.32)$$

where the matrix $[K_y]$ and the column vectors $\{v\}$ and $\{s(y)\}$ are given below,

$$
\underset{m \times m}{[K_y]} = \begin{bmatrix}
\underset{NZ \times NZ}{[K_{11}]} & \underset{NZ \times NZ}{[K_{12}]} & \underset{NZ \times NZ}{[K_{13}]} & \underset{NZ \times NZ}{[K_{14}]} & 0 & 0 & 0 \\
\underset{NZ \times NZ}{[K_{21}]} & \underset{NZ \times NZ}{[K_{22}]} & \underset{NZ \times NZ}{[K_{21}]} & \underset{NZ \times NZ}{[K_{14}]} & 0 & 0 & 0 \\
\underset{NZ \times NZ}{[K_{14}]} & \underset{NZ \times NZ}{[K_{21}]} & \underset{NZ \times NZ}{[K_{33}]} & \underset{NZ \times NZ}{[K_{21}]} & \underset{NZ \times NZ}{[K_{14}]} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \underset{NZ \times NZ}{[K_{14}]} & \underset{NZ \times NZ}{[K_{21}]} & \underset{NZ \times NZ}{[K_{33}]} & \underset{NZ \times NZ}{[K_{21}]} & \underset{NZ \times NZ}{[K_{14}]} \\
0 & 0 & 0 & \underset{NZ \times NZ}{[K_{14}]} & \underset{NZ \times NZ}{[K_{21}]} & \underset{NZ \times NZ}{[K_{22}]} & \underset{NZ \times NZ}{[K_{21}]} \\
0 & 0 & 0 & \underset{NZ \times NZ}{[K_{14}]} & \underset{NZ \times NZ}{[K_{13}]} & \underset{NZ \times NZ}{[K_{12}]} & \underset{NZ \times NZ}{[K_{11}]}
\end{bmatrix} \tag{2.33}
$$

$$
\underset{NZ \times NZ}{[K_{11}]} = \begin{bmatrix}
\frac{5C_1}{3h_z^2} + \frac{5C_1}{3h_x^2} & -\frac{8C_1}{3h_z^2} & -\frac{C_1}{6h_z^2} & 0 & 0 & 0 & 0 \\
-\frac{4C_1}{3h_z^2} & \frac{31C_1}{12h_z^2} + \frac{5C_1}{3h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{5C_1}{2h_z^2} + \frac{5C_1}{3h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{5C_1}{2h_z^2} + \frac{5C_1}{3h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} \\
0 & 0 & 0 & \frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{31C_1}{12h_z^2} + \frac{5C_1}{3h_x^2} & -\frac{4C_1}{3h_z^2} \\
0 & 0 & 0 & \frac{C_1}{12h_z^2} & -\frac{C_1}{3h_z^2} & -\frac{17C_1}{12h_z^2} & \frac{5C_1}{3h_z^2} + \frac{5C_1}{3h_x^2}
\end{bmatrix}
$$

$$
\underset{NZ \times NZ}{[K_{22}]} = \begin{bmatrix}
\frac{5C_1}{2h_z^2} + \frac{31C_1}{12h_x^2} & -\frac{8C_1}{3h_z^2} & -\frac{C_1}{6h_z^2} & 0 & 0 & 0 & 0 \\
-\frac{4C_1}{3h_z^2} & \frac{31C_1}{12h_z^2} + \frac{31C_1}{12h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{5C_1}{2h_z^2} + \frac{31C_1}{12h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{5C_1}{2h_z^2} + \frac{31C_1}{12h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} \\
0 & 0 & 0 & \frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{31C_1}{12h_z^2} + \frac{31C_1}{12h_x^2} & -\frac{4C_1}{3h_z^2} \\
0 & 0 & 0 & \frac{C_1}{12h_z^2} & -\frac{C_1}{3h_z^2} & -\frac{17C_1}{12h_z^2} & \frac{5C_1}{3h_z^2} + \frac{31C_1}{12h_x^2}
\end{bmatrix}
$$

$$
\underset{NZ\times NZ}{[K_{33}]} = \begin{bmatrix}
\frac{5C_1}{2h_z^2}+\frac{5C_1}{2h_x^2} & -\frac{8C_1}{3h_z^2} & \frac{C_1}{6h_z^2} & 0 & 0 & 0 & 0 \\
-\frac{4C_1}{3h_z^2} & \frac{31C_1}{12h_z^2}+\frac{5C_1}{2h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{5C_1}{2h_z^2}+\frac{5C_1}{2h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{5C_1}{2h_z^2}+\frac{5C_1}{2h_x^2} & -\frac{4C_1}{3h_z^2} & \frac{C_1}{12h_z^2} \\
0 & 0 & 0 & \frac{C_1}{12h_z^2} & -\frac{4C_1}{3h_z^2} & \frac{31C_1}{12h_z^2}+\frac{5C_1}{2h_x^2} & -\frac{4C_1}{3h_z^2} \\
0 & 0 & 0 & \frac{C_1}{12h_z^2} & -\frac{C_1}{3h_z^2} & -\frac{17C_1}{12h_z^2} & \frac{5C_1}{3h_z^2}+\frac{5C_1}{2h_x^2}
\end{bmatrix}
$$

$$[K_{12}] = -\frac{17C_1}{12h_x^2}[I]$$

$$[K_{13}] = -\frac{C_1}{3h_x^2}[I]$$

$$[K_{14}] = \frac{C_1}{12h_x^2}[I]$$

$$[K_{21}] = -\frac{4C_1}{3h_x^2}[I]$$

Matrix [I] is an identity matrix of order NZ x NZ. Note that C_1 is $\frac{1 - 2\nu}{2(1 - \nu)}$

$$\underset{m\times 1}{\{v\}} \quad \begin{bmatrix} \{v\}_1 \\ \{v\}_2 \\ \vdots \\ \{v\}_{NX-1} \\ \{v\}_{NX} \end{bmatrix} \qquad \underset{m\times 1}{\{s(y)\}} \quad \begin{bmatrix} \{s_1(y)\} \\ \{s_2(y)\} \\ \vdots \\ \{s_{NX-1}(y)\} \\ \{s_{NX}(y)\} \end{bmatrix} \tag{2.34}$$

where the partitioned column vectors are

$$\underset{NZx1}{\{v\}_1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{NZ-1} \\ v_{NZ} \end{bmatrix} \qquad \underset{NZx1}{\{v\}_2} = \begin{bmatrix} v_{NZ+1} \\ v_{NZ+2} \\ \vdots \\ v_{2NZ-1} \\ v_{2NZ} \end{bmatrix}$$

$$\underset{NZx1}{\{v\}} = \begin{bmatrix} v_{m-2NZ+1} \\ v_{m-2NZ+2} \\ \vdots \\ v_{m-NZ-1} \\ v_{m-NZ} \end{bmatrix} \qquad \underset{NZx1}{\{v\}_{NX}} = \begin{bmatrix} v_{m-NZ+1} \\ v_{m-NZ+2} \\ \vdots \\ v_{m-1} \\ v_m \end{bmatrix}$$

$$
\underset{NZx1}{\{s_1(y)\}} = \begin{bmatrix}
C_2\,(\dot{u}+\dot{w})_1 - \left.\dfrac{11C_1u}{6h_x}\right|_1 \\
C_2\,(\dot{u}+\dot{w})_2 - \left.\dfrac{11C_1u}{6h_x}\right|_2 \\
\vdots \\
C_2\,(\dot{u}+\dot{w})_{NZ-2} - \left.\dfrac{11C_1u}{6h_x}\right|_{NZ-2} \\
C_2\,(\dot{u}+\dot{w})_{NZ-1} - \left.\dfrac{11C_1u}{6h_x}\right|_{NZ-1} - \left.\dfrac{C_1w}{6h_z}\right|_{NZ} \\
C_2(\dot{u}+\dot{w})_{NZ} - \left.\dfrac{11C_1u}{6h_x}\right|_{NZ} + \left.\dfrac{11C_1w}{6h_z}\right|_{NZ}
\end{bmatrix}
$$

$$\{s_2(y)\}_{NZx1} = \begin{bmatrix} C_2\,(\dot{u}+\dot{w})_{NZ+1} + \left.\dfrac{C_1 u}{6h_x}\right|_1 \\ C_2\,(\dot{u}+\dot{w})_{NZ+2} + \left.\dfrac{C_1 u}{6h_x}\right|_2 \\ \vdots \\ C_2\,(\dot{u}+\dot{w})_{2NZ-2} + \left.\dfrac{C_1 u}{6h_x}\right|_{NZ-2} \\ C_2\,(\dot{u}+\dot{w})_{2NZ-1} + \left.\dfrac{C_1 u}{6h_x}\right|_{NZ-1} - \left.\dfrac{C_1 w}{6h_z}\right|_{2NZ} \\ C_2\,(\dot{u}+\dot{w})_{2NZ} + \left.\dfrac{C_1 u}{6h_x}\right|_{NZ} + \left.\dfrac{11C_1 w}{6h_z}\right|_{2NZ} \end{bmatrix}$$

$$\underset{NZx1}{\{s_3(y)\}} = \begin{bmatrix} C_2\ (\dot{u} + \dot{w})_{2NZ+1} \\ C_2\ (\dot{u} + \dot{w})_{2NZ+2} \\ \vdots \\ C_2\ (\dot{u} + \dot{w})_{3NZ-2} \\ C_2\ (\dot{u} + \dot{w})_{3NZ-1} - \dfrac{C_1 w}{6h_z}\Big|_{3NZ} \\ C_2\ (\dot{u} + \dot{w})_{3NZ} + \dfrac{11C_1 w}{6h_z}\Big|_{3NZ} \end{bmatrix}$$

$$\{s_{NX-1}(y)\}_{NZx1} = \begin{bmatrix} C_2\,(\dot{u}+\dot{w})_{m-2NZ+1} - \left.\frac{C_1 u}{6h_x}\right|_{m-NZ+1} \\ C_2\,(\dot{u}+\dot{w})_{m-2NZ+2} - \left.\frac{C_1 u}{6h_x}\right|_{m-NZ+2} \\ \vdots \\ C_2\,(\dot{u}+\dot{w})_{m-NZ-1} - \left.\frac{C_1 u}{6h_x}\right|_{m-1} - \left.\frac{C_1 w}{6h_z}\right|_{m-NZ} \\ C_2\,(\dot{u}+\dot{w})_{m-NZ} - \left.\frac{C_1 u}{6h_x}\right|_{m} + \left.\frac{11C_1 w}{6h_z}\right|_{m-NZ} \end{bmatrix}$$

$$\underset{NZ\times 1}{\{s_{NX}(y)\}} = \begin{bmatrix} C_2\,(\dot{u}+\dot{w})_{m-NZ+1} + \left.\dfrac{11C_1u}{6h_x}\right|_{m-NZ+1} \\ C_2\,(\dot{u}+\dot{w})_{m-NZ+2} + \left.\dfrac{11C_1u}{6h_x}\right|_{m-NZ+2} \\ \vdots \\ C_2\,(\dot{u}+\dot{w})_{m-2} + \left.\dfrac{11C_1u}{6h_x}\right|_{m-2} \\ C_2\,(\dot{u}+\dot{w})_{m-1} + \left.\dfrac{11C_1u}{6h_x}\right|_{m-1} - \left.\dfrac{C_1w}{6h_z}\right|_{m} \\ C_2\,(\dot{u}+\dot{w})_{m} + \left.\dfrac{11C_1u}{6h_x}\right|_{m} + \left.\dfrac{11C_1w}{6h_z}\right|_{m} \end{bmatrix}$$

2.2.3 Reduction of the Third Navier-Cauchy Equation and Associated Boundary Conditions for the Cracked Specimen

For the solution of equation (2.7), the lines parallel to z- axis in figure 3 are considered. The z- directional displacements of points along these lines will be denoted as $w_1, w_2, w_3, \ldots, w_n$. We define $\dot{u}|_1, \dot{u}|_2, \dot{u}|_3, \ldots, \dot{u}|_n$ as the derivatives of the x- directional displacements of the same points on these lines with respect to x and $\dot{v}|_1, \dot{v}|_2, \dot{v}|_3, \ldots, \dot{v}|_n$ as the derivatives of the y- directional displacements of the same points on these lines with respect to y. These displacements and derivatives can then be regarded as functions of z only since they are variables upon lines which are parallel to the z axis.

Following the procedure as before, the ordinary differential equation along a generic line ij may be written as,

$$\frac{d^2 w_{ij}}{dz^2} + C_1 \Big[\frac{1}{12h_y^2} \left(-w_{i-2,j} + 16\, w_{i-1,j} - 30\, w_{ij} + 16\, w_{i+1,j} - w_{i+2,j}\right)$$

$$+ \frac{1}{12h_x^2} \left(-w_{i,j-2} + 16\, w_{i,j-1} - 30\, w_{ij} + 16\, w_{i,j+1} - w_{i,j+2}\right)\Big]$$

$$= C_2 \frac{d}{dz} (\dot{u} + \dot{v})_{ij} \tag{2.35}$$

Similar differential equations are obtained for the other displacements of the points on the z- directional lines. Since each equation has the terms of the displacements of the points on the

surrounding lines, a system of ordinary differential equations is obtained for lines $w_1, w_2 \ldots, w_n$. The equation (2.35) is applicable only to interior lines. For boundary surface lines and lines adjacent to the boundary surface lines, the finite difference expressions for the second derivative will need the use of the fictitious lines. The displacements on these fictitious lines is eliminated with the help of shearing stresses on the free surfaces.

As shown in figure 8, the z- lines 1 through NX are split into two sets. The first set consists of lines 1 through NXC lying on the plane V(b). For these lines a fictitious line is used to write the finite difference equation, which is subsequently eliminated by using the shear stress boundary condition on this face. The second set consists of lines NXC + 1 through NX lying on the plane V(a). The w displacements are symmetric with respect to this plane. Therefore, symmetry consideration helps in eliminating displacements on the fictitious lines. A detailed description of the method used in deriving the equations for the lines lying on the boundary and adjacent to the boundary, can be found in the Appendix A. For example, the ordinary differential equation for line 1 is

$$\frac{d^2 w_1}{dz^2} + \frac{C_1}{12h_y^2} [-20\, w_1 + 17\, w_{NX+1} + 4\, w_{2NX+1} - w_{3NX+1}]$$

$$+ \frac{C_1}{12h_x^2} [-20\, w_1 + 17\, w_2 + 4\, w_3 - w_4]$$

$$= C_2 \frac{d}{dz} (\dot{u} + \dot{v})_1 - \frac{11C_1}{6h_x} \frac{du}{dz}\bigg|_1 - \frac{11C_1}{6h_y} \frac{dv}{dz}\bigg|_1 \tag{2.36}$$

For line NX - 1 the equation can be witten as:

$$\frac{d^2 w_{NX-1}}{dz^2} + \frac{C_1}{12h_y^2} [-30\, w_{NX-1} + 32\, w_{2NX-1} - 2\, w_{3NX-1}]$$

$$+ \frac{C_1}{12h_x^2} [-w_{NX-3} + 16\, w_{NX-2} - 31\, w_{NX-1} + 16\, w_{NX}]$$

$$= C_2 \frac{d}{dz} (\dot{u} + \dot{v})_{NX-1} - \frac{C_1}{6h_x} \frac{du}{dz}\bigg|_{NX} \tag{2.37}$$

Finally introducing matrix notation, all the ordinary differential equations along the z-directional lines are expressed in the form,

$$\underset{n\times 1}{\frac{d^2\{w\}}{dz^2}} = \underset{n\times n}{[K_z]}\ \underset{n\times 1}{\{w\}} + \underset{n\times 1}{\frac{d}{dz}\{t(z)\}} \tag{2.38}$$

where the matrix $[K_z]$ and the column vectors $\{w\}$ and $\{t(z)\}$ are given below,

$$
\underset{n\times n}{[K_z]} = \left[\begin{array}{c|c|c|c|c|c|c}
\underset{NX\times NX}{[K_{11}]} & \underset{NX\times NX}{[K_{12}]} & \underset{NX\times NX}{[K_{13}]} & \underset{NX\times NX}{[K_{14}]} & 0 & 0 & 0 \\
\hline
\underset{NX\times NX}{[K_{21}]} & \underset{NX\times NX}{[K_{22}]} & \underset{NX\times NX}{[K_{21}]} & \underset{NX\times NX}{[K_{24}]} & 0 & 0 & 0 \\
\hline
\underset{NX\times NX}{[K_{24}]} & \underset{NX\times NX}{[K_{21}]} & \underset{NX\times NX}{[K_{11}]} & \underset{NX\times NX}{[K_{21}]} & \underset{NX\times NX}{[K_{24}]} & 0 & 0 \\
\hline
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\hline
0 & 0 & \underset{NX\times NX}{[K_{24}]} & \underset{NX\times NX}{[K_{21}]} & \underset{NX\times NX}{[K_{33}]} & \underset{NX\times NX}{[K_{21}]} & \underset{NX\times NX}{[K_{24}]} \\
\hline
0 & 0 & 0 & \underset{NX\times NX}{[K_{24}]} & \underset{NX\times NX}{[K_{21}]} & \underset{NX\times NX}{[K_{22}]} & \underset{NX\times NX}{[K_{21}]} \\
\hline
0 & 0 & 0 & \underset{NX\times NX}{[K_{24}]} & \underset{NX\times NX}{[K_{NY,NY-2}]} & \underset{NX\times NX}{[K_{NY,NY-1}]} & \underset{NX\times NX}{[K_{NY,NY}]}
\end{array}\right] \tag{2.39}
$$

$$
[K_{ij}]_{NX \times NX} =
\left[
\begin{array}{c}
\begin{bmatrix}
\frac{5C_1}{[illegible]h_x^2}+\frac{5C_1}{3h_y^2} & -\frac{17C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2}+\frac{5C_1}{3h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2}+\frac{5C_1}{3h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2}+\frac{5C_1}{3h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0
\end{bmatrix}_{NXC,\,NY} \\
\begin{bmatrix}
0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2}+\frac{5C_1}{2h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2}+\frac{5C_1}{2h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2}+\frac{5C_1}{2h_y^2} & -\frac{4C_1}{3h_x^2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & -\frac{17C_1}{12h_x^2} & \frac{5C_1}{[illegible]h_x^2}+\frac{5C_1}{2h_y^2}
\end{bmatrix}_{NX-NXC,\,NX}
\end{array}
\right]
$$

$$
[K_{22}]_{NX \times NX} = \begin{bmatrix}
\frac{5C_1}{3h_x^2} + \frac{31C_1}{12h_y^2} & -\frac{17C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 \\
-\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2} + \frac{31C_1}{12h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2} + \frac{31C_1}{12h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2} + \frac{31C_1}{12h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} \\
0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2} + \frac{31C_1}{12h_y^2} & -\frac{4C_1}{3h_x^2} \\
0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & -\frac{17C_1}{12h_x^2} & \frac{5C_1}{3h_x^2} + \frac{31C_1}{12h_y^2}
\end{bmatrix}
$$

$$
\underset{NX \times NX}{[K_{33}]} =
\begin{bmatrix}
\frac{5C_1}{3h_x^2} + \frac{5C_1}{2h_y^2} & -\frac{17C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 \\
-\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2} + \frac{5C_1}{2h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2} + \frac{5C_1}{2h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2} + \frac{5C_1}{2h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} \\
0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2} + \frac{5C_1}{2h_y^2} & -\frac{4C_1}{3h_x^2} \\
0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & -\frac{17C_1}{12h_x^2} & \frac{5C_1}{3h_x^2} + \frac{5C_1}{2h_y^2}
\end{bmatrix}
$$

$$
[K_{NY,NY}]_{NX\times NX} = \begin{bmatrix}
\frac{5C_1}{3h_x^2} + \frac{5C_1}{3h_y^2} & -\frac{17C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 \\
-\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2} + \frac{5C_1}{3h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 & 0 \\
\frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2} + \frac{5C_1}{3h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{5C_1}{2h_x^2} + \frac{5C_1}{3h_y^2} & -\frac{4C_1}{3h_x^2} & \frac{C_1}{12h_x^2} \\
0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{4C_1}{3h_x^2} & \frac{31C_1}{12h_x^2} + \frac{5C_1}{3h_y^2} & -\frac{4C_1}{3h_x^2} \\
0 & 0 & 0 & \frac{C_1}{12h_x^2} & -\frac{C_1}{3h_x^2} & -\frac{17C_1}{12h_x^2} & \frac{5C_1}{3h_x^2} + \frac{5C_1}{3h_y^2}
\end{bmatrix}
$$

$$[K_{12}] = \left[\begin{array}{c|c} [K_{12}]' & [0] \\ \hline [0] & [K_{12}]'' \end{array}\right]$$

where
$$[K_{12}]' = -\frac{17C_1}{12h_y^2}[I]_{NXC,NXC}$$

$$[K_{12}]'' = -\frac{8C_1}{3h_y^2}[I]_{NX-NXC,\ NX-NXC}$$

$$[K_{13}] = \left[\begin{array}{c|c} [K_{13}]' & [0] \\ \hline [0] & [K_{13}]'' \end{array}\right]$$

$$[K_{13}]' = -\frac{C_1}{3h_y^2}[I]_{NXC,NXC}$$

$$[K_{13}]'' = \frac{C_1}{6h_y^2}[I]_{NX-NXC,\ NX-NXC}$$

$$\underset{NXxNX}{[K_{14}]} = \left[\begin{array}{c|c} [K_{14}]' & [0] \\ \hline [0] & [0] \end{array}\right]$$

$$[K_{14}]' = \frac{C_1}{12h_y^2}[I]_{NXC,NXC}$$

$$[K_{21}] = -\frac{4C_1}{3h_y^2}[I]_{NXxNX}$$

$$[K_{24}] = \frac{C_1}{12h_y^2}[I]_{NXxNX}$$

$$[K_{NY,NY-2}] = -\frac{C_1}{3h_y^2}[I]_{NXxNX}$$

$$[K_{NY,NY-1}] = -\frac{17}{12h_y^2}[I]_{NXxNX}$$

$$\underset{n\times 1}{\{w\}} = \begin{bmatrix} \{w\}_1 \\ \{w\}_2 \\ \vdots \\ \{w\}_{NY-1} \\ \{w\}_{NY} \end{bmatrix} \qquad \underset{n\times 1}{\{t(z)\}} = \begin{bmatrix} \{t_1(z)\} \\ \{t_2(z)\} \\ \vdots \\ \{t_{NY-1}(z)\} \\ \{t_{NY}(z)\} \end{bmatrix} \tag{2.40}$$

where the partitioned column vectors are

$$\underset{NX\times 1}{\{w\}_1} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{NX-1} \\ w_{NX} \end{bmatrix} \qquad \underset{NX\times 1}{\{w\}_2} = \begin{bmatrix} w_{NX+1} \\ w_{NX+2} \\ \vdots \\ w_{2NX-1} \\ w_{2NX} \end{bmatrix}$$

$$
\underset{NX\times 1}{(w)_{NY-1}} = \begin{bmatrix} w_{n-2NX+1} \\ w_{n-2NX+2} \\ \vdots \\ w_{n-NX-1} \\ w_{n-NX} \end{bmatrix}
\qquad
\underset{NX\times 1}{(w)_{NY}} = \begin{bmatrix} w_{n-NX+1} \\ w_{n-NX+2} \\ \vdots \\ w_{n-1} \\ w_{n} \end{bmatrix}
$$

$$
\underset{NX\times 1}{\{t_2(z)\}} = \left[\begin{array}{l}
\begin{bmatrix}
C_2\,(\dot{u}+\dot{v})_1 - \left.\dfrac{11C_1u}{6h_x}\right|_1 - \left.\dfrac{11C_1v}{6h_y}\right|_1 \\
C_2\,(\dot{u}+\dot{v})_2 + \left.\dfrac{C_1u}{6h_x}\right|_1 - \left.\dfrac{11C_1v}{6h_y}\right|_2 \\
C_2\,(\dot{u}+\dot{v})_3 - \left.\dfrac{11C_1v}{6h_y}\right|_3 \\
\vdots \\
C_2\,(\dot{u}+\dot{v})_{NXC} - \left.\dfrac{11C_1v}{6h_y}\right|_{NXC}
\end{bmatrix}_{NXC\times 1} \\
\hdashline
\begin{bmatrix}
C_2\,(\dot{u}+\dot{v})_{NXC+1} \\
\vdots \\
C_2\,(\dot{u}+\dot{v})_{NX-2} \\
C_2\,(\dot{u}+\dot{v})_{NX-1} - \left.\dfrac{C_1u}{6h_x}\right|_{NX} \\
C_2\,(\dot{u}+\dot{v})_{NX} + \left.\dfrac{11C_1u}{6h_x}\right|_{NX}
\end{bmatrix}_{(NX-NXC)\times 1}
\end{array}\right]
$$

$$\underset{NX\times 1}{\{t_2(z)\}} = \begin{bmatrix} \begin{bmatrix} C_2\,(\dot{u}+\dot{v})_1 - \left.\dfrac{11C_1u}{6h_x}\right|_{NX+1} + \left.\dfrac{C_1v}{6h_y}\right|_1 \\ C_2\,(\dot{u}+\dot{v})_2 + \left.\dfrac{C_1u}{6h_x}\right|_{NX+1} + \left.\dfrac{C_1v}{6h_y}\right|_2 \\ C_2\,(\dot{u}+\dot{v})_3 + \left.\dfrac{C_1v}{6h_y}\right|_3 \\ \vdots \\ C_2\,(\dot{u}+\dot{v})_{NXC} + \left.\dfrac{C_1v}{6h_y}\right|_{NXC} \end{bmatrix}_{NXC\times 1} \\ \hdashline \begin{bmatrix} C_2\,(\dot{u}+\dot{v})_{NXC+1} \\ \vdots \\ C_2\,(\dot{u}+\dot{v})_{NX-2} \\ C_2\,(\dot{u}+\dot{v})_{NX-1} - \left.\dfrac{C_1u}{6h_x}\right|_{2NX} \\ C_2\,(\dot{u}+\dot{v})_{NX} + \left.\dfrac{11C_1u}{6h_x}\right|_{2NX} \end{bmatrix}_{(NX-NXC)\times 1} \end{bmatrix}$$

$$
\underset{NX\times 1}{\{t_3(z)\}} = \begin{bmatrix}
C_2\,(\dot{u}+\dot{v})_{2NX+1} - \left.\dfrac{11C_1u}{6h_x}\right|_{2NX+1} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{2NX+2} + \left.\dfrac{C_1u}{6h_x}\right|_{2NX+1} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{2NX+3} \\
\vdots \\
C_2\,(\dot{u}+\dot{v})_{3NX-2} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{3NX-1} - \left.\dfrac{C_1u}{6h_x}\right|_{3NX} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{3NX} + \left.\dfrac{11C_1u}{6h_x}\right|_{3NX}
\end{bmatrix}
$$

$$
\underset{NX\times 1}{\{t_{NY-1}(z)\}} = \begin{bmatrix}
C_2\,(\dot{u}+\dot{v})_{n-2NX+1} - \dfrac{11C_1u}{6h_x}\Big|_{n-2NX+1} - \dfrac{C_1v}{6h_y}\Big|_{n-NX+1} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{n-2NX+2} + \dfrac{C_1u}{6h_x}\Big|_{n-2NX+1} - \dfrac{C_1v}{6h_y}\Big|_{n-NX+2} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{n-2NX+3} - \dfrac{C_1v}{6h_y}\Big|_{n-NX+3} \\
\vdots \\
C_2\,(\dot{u}+\dot{v})_{n-NX-2} - \dfrac{C_1v}{6h_y}\Big|_{n-2} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{n-NX-1} - \dfrac{C_1u}{6h_x}\Big|_{n-NX} - \dfrac{C_1v}{6h_y}\Big|_{n-1} \\[2ex]
C_2\,(\dot{u}+\dot{v})_{n-NX} + \dfrac{11C_1u}{6h_x}\Big|_{n-NX} - \dfrac{C_1v}{6h_y}\Big|_{n}
\end{bmatrix}
$$

$$
\underset{NX\times 1}{\{t_{NY}(z)\}} = \begin{bmatrix}
C_2\ (\dot{u} + \dot{v})_{n-NX+1} - \left.\dfrac{11C_1u}{6h_x}\right|_{n-NX+1} + \left.\dfrac{11C_1v}{6h_y}\right|_{n-NX+1} \\
C_2\ (\dot{u} + \dot{v})_{n-NX+2} + \left.\dfrac{C_1u}{6h_x}\right|_{n-NX+1} + \left.\dfrac{11C_1v}{6h_y}\right|_{n-NX+2} \\
C_2\ (\dot{u} + \dot{v})_{n-NX+3} + \left.\dfrac{11C_1v}{6h_y}\right|_{n-NX+3} \\
\vdots \\
C_2\ (\dot{u} + \dot{v})_{n-2} + \left.\dfrac{11C_1v}{6h_y}\right|_{n-2} \\
C_2\ (\dot{u} + \dot{v})_{n-1} - \left.\dfrac{C_1u}{6h_x}\right|_{n} + \left.\dfrac{11C_1v}{6h_y}\right|_{n-1} \\
C_2\ (\dot{u} + \dot{v})_{n} + \left.\dfrac{11C_1u}{6h_x}\right|_{n} + \left.\dfrac{11C_1v}{6h_y}\right|_{n}
\end{bmatrix}
$$

2.3 Solution of Differential Equations with Constant Coefficients

The set of ℓ second order differential equations represented by equation (2.22) can be reduced to a set of 2ℓ first order differential equations by treating the derivatives of the u's as an additional set of ℓ unknowns, that is, defining

$$u_{\ell+1} = \frac{du_1}{dx}, \quad u_{\ell+2} = \frac{du_2}{dx} \quad \text{etc.}$$

The resulting 2ℓ equations can now be written as a single first order matrix differential equation

$$\frac{dU}{dx} = A_1 U + \frac{dR(x)}{dx} \tag{2.41}$$

where U and R are column matrices of 2ℓ each and A_1 is $2\ell \times 2\ell$ matrix of constant coefficients and are written as follows,

$$\underset{2\ell x1}{U} = \left[\begin{array}{c} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_\ell \end{bmatrix}_{\ell x1} \\ \hdashline \begin{bmatrix} \frac{du_1}{dx} \\ \frac{du_2}{dx} \\ \vdots \\ \frac{du_\ell}{dx} \end{bmatrix}_{\ell x1} \end{array} \right] \tag{2.42}$$

$$\underset{2\ell x1}{R} = \begin{bmatrix} \{0\}_{\ell x1} \\ \hdashline \{r(x)\}_{\ell x1} \end{bmatrix} \tag{2.43}$$

and

$$\underset{2\ell x2\ell}{A_1} = \left[\begin{array}{c:c} [0]_{\ell x\ell} & [I]_{\ell x\ell} \\ \hdashline [K_x]_{\ell x\ell} & [0]_{\ell x\ell} \end{array}\right] \tag{2.44}$$

In the similar manner the second order ordinary differential equation given by eqution (2.32) can be transformed to a first order ordinary differential equation and it is given by,

$$\frac{dV}{dy} = A_2\ V + \frac{dS(y)}{dy} \tag{2.45}$$

where matrix A_2 and column vectors V and S(y) are

$$\underset{2mx1}{V} = \begin{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}_{mx1} \\ \hdashline \begin{bmatrix} \frac{dv_1}{dx} \\ \frac{dv_2}{dx} \\ \frac{dv_m}{dx} \end{bmatrix}_{mx1} \end{bmatrix} \tag{2.46}$$

$$\underset{2m\times1}{S(y)} = \left[\begin{array}{c} \{0\}_{m\times1} \\ \hdashline \{s(y)\}_{m\times1} \end{array}\right] \tag{2.47}$$

and

$$\underset{2m\times2m}{A_2} = \left[\begin{array}{c:c} [0]_{m\times m} & [I]_{m\times m} \\ \hdashline [K_y]_{m\times m} & [0]_{m\times m} \end{array}\right] \tag{2.48}$$

For the z- directional lines the second order differential equation given by equation (2.38) is also transformed to a first order ordinary differential equation. The transformed equation is,

$$\frac{dW'}{dz} = A_3\, W' + \frac{dT(z)}{dz} \tag{2.49}$$

where matrix A_3 and column vectors W and T(z) are given below,

$$\underset{2n\times1}{W'} = \left[\begin{array}{c} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}_{n\times1} \\ \hdashline \begin{bmatrix} \frac{dw_1}{dx} \\ \frac{dw_2}{dx} \\ \vdots \\ \frac{dw_n}{dx} \end{bmatrix}_{n\times1} \end{array}\right] \tag{2.50}$$

$$\underset{2n\times 1}{T(z)} = \left[\begin{array}{c} \{0\} \\ \hline \{t(z)\} \end{array}\right] \tag{2.51}$$

$$\underset{n\times n}{A_3} = \left[\begin{array}{c:c} [0]_{n\times n} & [I]_{n\times n} \\ \hdashline [K_z]_{n\times n} & [0]_{n\times n} \end{array}\right] \tag{2.52}$$

2.3.1 Solution by Recurrence Relations

The system of ordinary differential equations (2.41), (2.45) and (2.49) can be solved by any number of standard techniques. An algorithm is derived for obtaining the solution by using a recurrence relation to sweep through, from one boundary to the opposite one, once, to obtain the missing boundary conditions and then to sweep through a second time to obtain the complete solution everywhere. These equations are not independent of each other. They are coupled to the other equation through the vectors R, S, and T. Therefore, the solution of the equations can be achieved by an iterative technique in which the solution for two sets of different directional lines is used to evaluate the coupling terms for the third set of directional lines. Then the complete solution for this set of directional lines is achieved through the recurrence relation method described below.

Consider the matrix equation

$$\frac{dU}{dx} = AU + \frac{dR}{dx} \tag{2.53}$$

Let the x lines be divided into n intervals (not necessarily equally spaced) with h_i the spacing between the (i-1)st and ith nodal points. Then

$$\frac{U_i - U_{i-1}}{h_i} = \frac{A_i U_i + A_{i-1} U_{i-1}}{2} + \frac{R_i - R_{i-1}}{h_i}$$

$$(I - \frac{h_i}{2} A_i) U_i = (I + \frac{h_i}{2} A_{i-1}) U_{i-1} + (R_i - R_{i-1}) \tag{2.54}$$

or

$$U_i = (I - \frac{h_i}{2} A_i)^{-1} (I + \frac{h_i}{2} A_{i-1})U_{i-1}$$

$$+ (I - \frac{h_i}{2} A_i)^{-1} (R_i - R_{i-1}) \tag{2.55}$$

or

$$U_i = L_i U_{i-1} + M_i$$

where

$$L_i = (I - \frac{h_i}{2} A_i)^{-1} (I + \frac{h_i}{2} A_{i-1})$$

$$M_i = (I - \frac{h_i}{2} A_i)^{-1} (R_i - R_{i-1}) \tag{2.56}$$

Let

$$U_i = D_i U_1 + F_i \tag{2.57}$$

Then

$$D_i U_1 + F_i = L_i U_{i-1} + M_i$$

$$= L_i (D_{i-1} U_1 + F_{i-1}) + M_i$$

or

$$(D_i - L_i D_{i-1}) U_1 = L_i F_{i-1} - F_i + M_i$$

Since U_1 is abritraty

$$D_i = L_i D_{i-1}$$

$$F_i = L_i F_{i-1} + M_i \tag{2.58}$$

For station 2, we can write from (2.57)

$$U_2 = D_2 U_1 + F_2$$

and from (2.55)

$$U_2 = L_2 U_1 + M_2$$

$$\therefore \quad D_2 = L_2, \; F_2 = M_2 \tag{2.59}$$

All the D_i and F_i can now be determined from 2.58, starting with (2.59)

At the last station, i = n, we have

$$U_n = D_n U_1 + F_n \tag{2.60}$$

Now parts of the vectors U_1 and U_n are known (usually half the boundary conditions will be given at each end). Let us write these vectors as

$$U_1 = \begin{bmatrix} U_{1,k} \\ U_{1,u} \end{bmatrix}, \qquad U_n = \begin{bmatrix} U_{n,k} \\ U_{n,u} \end{bmatrix} \tag{2.61}$$

where k represents the known part of the vector and u the unknown part. Let us further partition D_n and F_n as

$$D_n = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \qquad F_n = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \tag{2.62}$$

So that we can write

$$\begin{bmatrix} U_{n,k} \\ U_{n,u} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{bmatrix} U_{1,k} \\ U_{1,u} \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \tag{2.63}$$

Then

$$U_{n,k} = \alpha_1 U_{1,k} + \alpha_2 U_{1,u} + \beta_1$$

or

$$U_{1,u} = \alpha_2^{-1} \left(U_{n,k} - \alpha_1 U_{1,k} - \beta_1\right) \tag{2.64}$$

The vector U_1 is now completely known and the complete solution obtained.

It is apparent that if the first part of the vector U_1 is the unknown part and/or the first part of U_n is unknown, then the appropriate relation, taken from equation (2.63), must be used to determine $U_{1,u}$.

Thus, suppose the first half of U_1 is unknown and the second half known, so that

$$U_1 = \begin{bmatrix} U_{1,u} \\ U_{1,k} \end{bmatrix}$$

Then from eq. (2.63)

$$U_{n,k} = \alpha_1 U_{1,u} + \alpha_2 U_{1,k} + \beta_1$$

and

$$U_{1,u} = \alpha_1^{-1} \left(U_{n,k} - \alpha_2 U_{1,k} - \beta_1\right) \tag{2.65}$$

similarly if the first part of U_n is unknown and the second part is known, then

$$U_n = \begin{bmatrix} U_{n,u} \\ U_{n,k} \end{bmatrix}$$

and

$$U_{n,k} = \alpha_3\, U_{1,k} + \alpha_4\, U_{1,u} + \beta_2$$

and

$$U_{1,u} = \alpha_4^{-1}\,(U_{n,k} - \alpha_3\, U_{1,k} - \beta_2) \tag{2.66}$$

and finally if $U_1 = \begin{bmatrix} U_{1,u} \\ U_{1,k} \end{bmatrix}$ and $U_n = \begin{bmatrix} U_{n,u} \\ U_{n,k} \end{bmatrix}$

$$U_{n,k} = \alpha_3\, U_{1,u} + \alpha_4\, U_{1,k} + \beta_2$$

and

$$U_{1,u} = \alpha_3^{-1}\,(U_{n,k} - \alpha_4\, U_{1,k} - \beta_2) \tag{2.67}$$

To summarize:

1. The vectors A_i and R_i are known (R obtained by iteration). If the spacing between lines is maintained constant, then $A_i \equiv A$ is a constant matrix.

2. Calculate the L_i and M_i from (2.56).

3. Calculate the D_i and F_i from (2.59) and (2.58).

4. Partition the matrices D_n and F_n as indicated by equations (2.62) and (2.63). Note that if there are M lines then the vector U will have 2M elements and there will be 2M boundary conditions, divided between stations 1 and n. This will determine the length of the known and unknown vectors and thus the dimensions of the α's and β's.

5. Calculate the unknown vector $U_{1,u}$ from (2.64) or (2.65-2.67). The complete vector U_1 is now known.

6. The displacements at any station are now computed from (2.57). As noted above if the spacing between lines is constant, i.e., h_x, h_y, and h_z are constants, then A is a constant matrix. It then follows from (2.56) that $L_i = L$ is a constant matrix (if h_i is also constant). Then equations (2.58) read

$$D_i = LD_{i-1}$$

$$F_i = LF_{i-1} + M_i$$

Note that although M_i varies along the lines, it is only a column matrix.

2.3.2 Incorporation of Prescribed Boundary Conditions of Normal Stresses or Applied Displacements in the Recurrence Relations.

The second order differential equations given by equation (2.22) for the x-lines involve a two point boundary value problem. The first half of the boundary conditions are given on face IV and the remaining boundary conditions are given on face I as shown in Figure 4. In the case of an edge crack specimen they are

$$\sigma_{xx}\big|_{IV} = 0 \qquad (2.68)$$

and $$\sigma_{xx}|_I = 0 \tag{2.69}$$

The subscripts IV and I refer to the boundary planes as shown in Figure 4. With the help of stress-displacement relations the equations (2.68) and (2.69) are reduced to the following equations respectively,

$$\frac{du}{dx}\Big|_{IV} = -\frac{\nu}{1-\nu}(\dot{v} + \dot{w})_{IV} \tag{2.70}$$

and $$\frac{du}{dx}\Big|_{I} = -\frac{\nu}{1-\nu}(\dot{v} + \dot{w})_{I} \tag{2.71}$$

Since the derivatives of v and w will be known on all the points on the face IV and I, a vector of $\{U_{1,k}\}_{\ell x1}$ and $\{U_{n,k}\}_{\ell x1}$ can be formed by assembling the derivatives of u in the proper order, as designated in Figure 6. These vectors will constitute the lower half of the complete vector used in the recurrence relation method. The complete set of vectors are

$$\underset{2\ell x1}{U_1} = \begin{bmatrix} \{U_{1,u}\}_{\ell x1} \\ \{U_{1.k}\}_{\ell x1} \end{bmatrix} \tag{2.72}$$

and $$\underset{2\ell x\ell}{U_n} = \begin{bmatrix} \{U_{n,u}\}_{\ell x1} \\ \{U_{n,k}\}_{\ell x1} \end{bmatrix} \tag{2.73}$$

As described in the previous section using equation (2.66) the remaining half of the vector U_1 can be calculated and a forward sweep is made to obtain the solution at every grid point. This procedure presents a

problem because both the boundary conditions are Neumann's type and due to this the solution can be determined only upto an arbitrary constant. The presence of rigid body displacements makes matrix α_3 in equation (2.66), a singular matrix. To suppress the rigid body displacement, the displacement $u_\ell|_I$ is set to zero. This leads to a rearrangement of matrices α_3 and α_4 in equation (2.66). Once $\{U_{1,u}\}_{\ell x1}$ is determined by using the rearranged equations, a forward sweep is carried out to determine the solution at all the grid points. The known value of $u_{2\ell}|_I$ is saved before the sweep is carried out. To maintain the consistancy the newly calculated value of $u_{2\ell}|_I$ is replaced by its old value.

The second order differential equation given by equation (2.32) for the set of y-directional lines is again a two point boundary value problem. Half of the boundary conditions are prescribed on face V and the remaining half of boundary conditions are given on face II. The face V and face II are the boundary planes as shown in Figure 4. It also shows the cracked face designated by the plane V(b). This leads to two kinds of boundary conditions for the points lying on the plane V. The face V(b) is a traction free surface, therefore all the y- lines, starting from this face will satisfy the following boundary condition,

$$\sigma_{yy}|_{V(b)} = 0 \tag{2.74}$$

Due to the symmetry of v-displacements with respect to the plane V(a), all the lines starting from this face will satisfy the condition,

$$v\big|_{V(b)} = 0 \tag{2.75}$$

using the stress-displacement relations equation (2.74) can be reduced to

$$\frac{dv}{dy}\bigg|_{V(b)} = - \frac{\nu}{1-\nu} \left(\frac{du}{dx} + \frac{dw}{dz}\right)_{V(b)} \tag{2.76}$$

The equations (2.75) and (2.76) are assembled into a vector form by following the sequence of numering of the y-directional lines as shown in Figure 7. The assembled vectors are $V'_{1,k}$ and $V''_{1,k}$. Using these vectors, the vector V_1 at face V is written as

$$\underset{2m\times 1}{V_1} = \begin{bmatrix} V'_{1,u} \\ V'_{1,k} \\ V''_{1,k} \\ V''_{1,u} \end{bmatrix}$$

where $V'_{1\ u}$ and $V''_{1,u}$ represent the unknown parts of the vector V_1. Rest of the boundary conditions are obtained by utilizing normal traction at the face II. As shown in Figure 2, in the case of tensile loading we have an applied normal traction σ on this face. Using the stress-displacement relation, we can write,

$$\left.\frac{dv}{dy}\right|_{II} = \left\{\frac{\sigma}{E}\,\frac{(1+\nu)\,(1-2\nu)}{(1-\nu)}\right\}_{II} - \frac{\nu}{(1-\nu)}\left(\frac{du}{dx}+\frac{dw}{dz}\right)_{II} \tag{2.77}$$

The equation (2.77) is arranged to form the vector $V_{n,k}$. The vector V_n at face II can now be written as

$$V_n = \begin{bmatrix} V_{n,u} \\ V_{n,k} \end{bmatrix} \tag{2.78}$$

$V_{n,u}$ represents the unknown part of the vector V_n. Following the recurrence relation method, an equation similar to the equation (2.57) is formulated and given by

$$V_n = D_n\, V_1 + F_n \tag{2.79}$$

The evaluation of matrices D_n and F_n is described in the previous section 2.3.1. The determination of unknown vectors $V'_{1,u}$ and $V''_{1,u}$ differs slightly from the procedure given in the section 2.3.1. The details of the procedure used to calculate vector V_1 can be found in Appendix B. Once the complete vector V_1 is known, using the recurrence relation method, the displacements v and $\dot{v}$ are obtained at all the grid points.

The second order ordinary differential equation for the set of z- directional lines given by the equation (2.38) is a two point boundary value problem. As shown in Figure 4, half of the boundary conditions are obtained by setting all the w- displacements equal to zero on the plane VI due to symmetry. This leads to

$$\underset{n\times 1}{W_{1,k}} = 0 \qquad (2.80)$$

on the boundary plane III, the prescribed normal tractions are zero. This leads to a set of equations for each grid point on this plane. These equations in vector form can be written as

$$\underset{n\times 1}{W_{n,k}} = \{\frac{dw}{dz}\}_{n\times 1} = -\frac{\nu}{1-\nu} [\{\dot{u}\}_{n\times 1} + \{\dot{v}\}_{n\times 1}] \qquad (2.81)$$

once again the vectors W_1 and W_n are formed as

$$\underset{2n\times 1}{W_1} = \begin{bmatrix} W_{1,k} \\ W_{1,u} \end{bmatrix}$$

and

$$W_n = \begin{bmatrix} W_{n,u} \\ W_{n,k} \end{bmatrix} \qquad (2.82)$$

the general equation connecting the vectors W_1 and W_n is

$$W_n = D_n W_1 + F_n \qquad (2.\text{..})$$

The matrices D_n and F_n can be partitioned as shown in equation (2.63). Appropriate matrix equations are extracted to solve for $W_{1,u}$. The complete solution for w- displacements and its derivative $\dot{w}$ is obtained

for all the grid points by carrying out a forward sweep operation, described in section 2.3.1.

2.4 Loading Idealization for Compact Tension Specimen

One of the major objectives of this investigation is to study the compact tension specimen. Figure 1(a) shows a standard compact tension specimen and the loading on it. The idealized model of this specimen used in the present study is shown in Figure 1(b). The pin loading applied on the specimen is approximated by a parabolically varying shear type loading. The resultant force due to the applied shear loading is maintained equal to the actual applied pin load (25).

This type of prescribed shear loading can not be incorporated into the solution of the ordinary differential equations in which the normal traction boundary conditions have been used. The equations for y- lines are slightly modified to include this type of traction boundary condition. The y- directional lines whose equations are directly affected by this are 1, 2, . . NZ and NZ + 1, NZ + 2 . . 2NZ.

The traction σ_{ny} at the boundary face IV is

$$\sigma_{ny}\Big|_{IV} = -\tau(y) \tag{2.84}$$

where $\tau(y)$ is the prescribed surface traction. Its variation in y-direction is expressed as

$$\tau(y) = 4\ \tau_o\left(\frac{y}{h}\right)\ \left(1 - \frac{y}{h}\right) \tag{2.85}$$

τ_o is the maximum value of prescribed surface traction at $y = \frac{h}{2}$ and h is the semi length of the specimen as shown in Figure 1(b). The

inclusion of this leads to a new equation for the y- directional line 1. The modified equation is

$$\frac{d^2 v_1}{dy^2} + \frac{C_1}{12h_x^2} [- 20v_1 + 17v_{NZ+1} + 4v_{2NZ+1} - v_{3NZ+1}]$$

$$+ \frac{C_1}{12h_z^2} [- 30v_1 + 32V_2 - 2V_3]$$

$$= C_2 \frac{d}{dy} (\dot{u} + \dot{w})_1 - \frac{11C_1}{6h_x} \frac{d\text{[illegible]}}{dy} \text{[illegible]} \frac{\text{[illegible]}_1}{\text{[illegible]}} [\frac{\tau(y)}{G}] \tag{2.86}$$

where G is the shear modulus of the material. Similarly the equations for lines 2 through NZ can be written. The equation (2.86) is the same as the equation (2.30) with the additional term of $- \frac{11C_1}{6h_x} [\frac{\tau(y)}{G}]$, which appears due to a non-zero prescribed surface traction. The modified equations for line 2 through NZ also include this additional term. The other terms for these equations remain the same as before. The ordinary differential equations for the line NZ+1 through 2NZ are modified by adding additional term of $\frac{C_1}{6h_x} \frac{\tau(y)}{G}$ to the original equations. The details of the method used to derive these equations can be found in the Appendix C.

Introducing the matrix notation, all the ordinary differential equations along the y- directional lines are expressed in the form,

$$\underset{m\times 1}{\frac{d^2\{v\}}{dy^2}} = \underset{m\times m}{[K_y]}\ \underset{m\times 1}{\{v\}} + \underset{m\times 1}{\frac{d}{dy}\{s(y)\}} + \underset{m\times 1}{\{s^*(y)\}} \tag{2.87}$$

The equation (2.87) is similar to the equation (2.32) with an addition of a new vector $\{s^*(y)\}$ whose components are

$$\underset{m\times 1}{\{s^*(y)\}} = \begin{bmatrix} \{s_1^*(y)\}_{NZ\times 1} \\ \\ \{s_2^*(y)\}_{NZ\times 1} \\ \\ \{0\}_{m-2NZ} \end{bmatrix} \tag{2.88}$$

where

$$\underset{NZ\times 1}{\{s_1^*(y)\}} = -\frac{11C_1}{6h_x}\,\frac{\tau(y)}{G}\begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

and

$$\underset{NZ\times 1}{\{s_2^*(y)\}} = \frac{C_1}{6h_x}\,\frac{\tau(y)}{G}\begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

once again by following the procedure described in section 2.3, the equations (2.87) are reduced to a set of first order ordinary differential equations and the new set of equations are,

$$\frac{dV}{dy} = A_2 V + \frac{dS(y)}{dy} + S^*(y) \tag{2.89}$$

Definitions of vectors V, S(y) and matrix A_2 are given by equations (2.46), (2.47) and (2.48) respectively. The vector $S^*(y)$ is a new addition and defined as

$$\underset{2m\mathrm{x}1}{S^*(y)} = \begin{bmatrix} \{0\}_{m\mathrm{x}1} \\ \{s^*(y)\}_{m\mathrm{x}1} \end{bmatrix}$$

The equation (2.89) can not be solved by the direct application of the method of recurrence relation described in section 2.3.1. It will require some small modification due to presence of vector $S^*(y)$.

Let the y lines be divided into n intervals with h_i being the spacing between the (i-1)st and the ith nodal points. Then,

$$\frac{V_i - V_{i-1}}{h_i} = \frac{A_i V_i + A_{i-1} V_{i-1}}{2} + \frac{S_i - S_{i-1}}{h_i} + \frac{S_i^* + S_{i-1}^*}{2} \tag{2.90}$$

The above equation can be rearranged as

$$V_i = [I - \frac{h_i}{2} A_i]^{-1} [I + \frac{h_i}{2} A_{i-1}] V_{i-1}$$

$$+ [I - \frac{h_i}{2} A_i]^{-1} [S_i - S_{i-1} + \frac{h_i}{2} (S_i^* + S_{i-1}^*)]$$

$$= L_i V_i + M_i$$

where

$$L_i = [I - \frac{h_i}{2} A_i]^{-1} [I + \frac{h_i}{2} A_{i-1}]$$

$$M_i = [I - \frac{h_i}{2} A_i]^{-1} [S_i - S_{i-1} + \frac{h_i}{2} (S_i^* + S_{i-1}^*)] \qquad (2.91)$$

we see an appearance of new terms in the M_i vector. Once these modified M_i vectors are calculated, the procedure described in the section (2.3.1) is followed.

CHAPTER 3

SOLUTION OF ELASTO-PLASTIC PROBLEM USING THE METHOD OF LINES

In this Chapter a three-dimensional elasto-plastic material behavior formulation for the governing field equations in incremental form is presented. The method of lines is extended to include the non-linear plastic strain terms. The solution procedure is specialized for the two geometries considered herein, but can easily be extented to other geometries.

3.1 Governing Field Equations

The material is assumed to be isotropic and homogeneous. The deformations are considered to be small and quasi-static. The Von Mises criterion given below is used to determine the yield condition at each material point.

$$\sigma_e = \frac{1}{\sqrt{2}} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6 (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)]^{1/2} \quad (3.1)$$

Subsequent yield surfaces are determined by using the rule of isotropic hardening. Prantl-Reuss relations are employed for constitutive equations relating the plastic strain rates and stresses.

In the plastic range the strains are in general not uniquely determined by the stresses, but depend on the whole history of loading.

Therefore, the load is applied in increments as fractions of the total loading and the equations are written in the incremental form. Let ε_{ij} be the total strain after the kth loading step. Following the conventional tensor notation it can be written as

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} + \Delta\varepsilon_{ij}^{p} \qquad i,j = 1,2,3 \tag{3.2}$$

where ε_{ij}^{e} is the strain due to the elastic deformations, ε_{ij}^{p} is the accumulated plastic strain through the (k-1)th loading step and $\Delta\varepsilon_{ij}^{p}$ is the plastic strain due to plastic flow in the kth increment. The equation (3.2) is rearranged as

$$\varepsilon_{ij}^{e} = \varepsilon_{ij} - \varepsilon_{ij}^{p} - \Delta\varepsilon_{ij}^{p} \tag{3.3}$$

The elastic strains are related to stresses in the following manner,

$$\varepsilon_{ij}^{e} = \frac{\sigma_{ij}}{2G} - \delta_{ij} \frac{\nu}{E} \sigma_{kk} \tag{3.4}$$

The incompressibility condition in tensor notation is

$$\varepsilon_{ii}^{p} = 0 \tag{3.5}$$

and

$$\Delta\varepsilon_{ii}^{p} = 0$$

(use of the double subscript represents the summation over that subscript.)

on making use of the equation (3.4) and (3.5), the equation (3.2) can

be modified as,

$$\sigma_{ij} = (\frac{E}{1 + \nu})\ \varepsilon_{ij} + \delta_{ij}\ (\frac{\nu}{1 + \nu})\ (\frac{E}{1 - 2\nu})\ \varepsilon_{kk} - (\frac{E}{1 + \nu})\ (\varepsilon^{p}_{ij} + \Delta\varepsilon^{p}_{ij}) \tag{3.6}$$

For small deformations, strain-displacement relations are,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.7}$$

Introducing equation (3.7) into equation (3.6) leads to

$$\sigma_{ij} = \frac{E}{2(1 - \nu)}(u_{i,j} + u_{j,i}) + \delta_{ij}\ (\frac{\nu}{1 + \nu})\ (\frac{E}{1 - 2\nu})\ u_{k,k} - (\frac{E}{1 + \nu})\ (\varepsilon^{p}_{ij} + \Delta\varepsilon^{p}_{ij}) \tag{3.8}$$

The equation (3.8) represents the constitutive relations for the elaso-plastic materials. The equilibrium equations for zero body force are

$$\sigma_{ij,i} = 0 \tag{3.9}$$

A set of equations similar to the Navier-Cauchy equations for the elasto-plastic material are obtained on substituting equation (3.8) into the equations (3.9). These equations are,

$$u_{j,ii} + \frac{1}{1 - 2\nu}\ u_{k,kj} = 2(\varepsilon^{p}_{ij,i} + \Delta\varepsilon^{p}_{ij,i}) \tag{3.10}$$

The equation (3.10) is essentially similar to the equation (2.4) except the addition of the plastic strain terms on the right hand side. For convenience the following non-dimensional quantities are introduced,

$$\bar{x} = \frac{x}{B}$$

$$\bar{y} = \frac{y}{B} \tag{3.11}$$

$$\bar{z} = \frac{z}{B}$$

$$\bar{\varepsilon}_{xx} = \frac{\varepsilon_{xx}}{\varepsilon_o} \qquad \bar{\varepsilon}_{xy} = \frac{\varepsilon_{xy}}{\varepsilon_o}$$

$$\bar{\varepsilon}_{yy} = \frac{\varepsilon_{yy}}{\varepsilon_o} \qquad \bar{\varepsilon}_{yz} = \frac{\varepsilon_{yz}}{\varepsilon_o} \tag{3.12}$$

$$\bar{\varepsilon}_{zz} = \frac{\varepsilon_{zz}}{\varepsilon_o} \qquad \bar{\varepsilon}_{zx} = \frac{\varepsilon_{zx}}{\varepsilon_o}$$

where $\varepsilon_o = \frac{\sigma_o}{E}$

E is the Young's modulus and σ_o is the yield stress of the material.

$$\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\sigma_o} \qquad \bar{\sigma}_{xy} = \frac{\sigma_{xy}}{\sigma_o}$$

$$\bar{\sigma}_{yy} = \frac{\sigma_{yy}}{\sigma_o} \qquad \bar{\sigma}_{yz} = \frac{\sigma_{yz}}{\sigma_o} \tag{3.13}$$

$$\bar{\sigma}_{zz} = \frac{\sigma_{zz}}{\sigma_o} \qquad \bar{\sigma}_{zx} = \frac{\sigma_{zx}}{\sigma_o}$$

and

$$\bar{u} = \frac{Eu}{\sigma_o B}$$

$$\bar{v} = \frac{Ev}{\sigma_o B} \qquad (3.14)$$

$$\bar{w} = \frac{Ew}{\sigma_o B}$$

where B is the thickness of the crack specimen. The substitution of non-dimensional quantities $\bar{\varepsilon}_{ij}$, $\bar{\sigma}_{ij}$ and $\bar{u}_i$ into (3.8) and (3.10) leads to the following non-dimensional equations,

$$\bar{\sigma}_{xx} = \frac{1}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial \bar{u}}{\partial \bar{x}} + \nu\frac{\partial \bar{v}}{\partial \bar{y}} + \nu\frac{\partial \bar{w}}{\partial \bar{z}}\right] - \left(\frac{1}{1+\nu}\right)(\bar{\varepsilon}^P_{xx} + \Delta\bar{\varepsilon}^P_{xx}) \qquad (3.15)$$

$$\bar{\sigma}_{yy} = \frac{1}{(1+\nu)(1-2\nu)}\left[\nu\frac{\partial \bar{u}}{\partial \bar{x}} + (1-\nu)\frac{\partial \bar{v}}{\partial \bar{y}} + \nu\frac{\partial \bar{w}}{\partial \bar{z}}\right] - \frac{1}{(1+\nu)}(\bar{\varepsilon}^P_{yy} + \Delta\bar{\varepsilon}^P_{yy}) \qquad (3.16)$$

$$\bar{\sigma}_{zz} = \frac{1}{(1+\nu)(1-2\nu)}\left[\nu\frac{\partial \bar{u}}{\partial \bar{x}} + \nu\frac{\partial \bar{v}}{\partial \bar{x}} + (1-\nu)\frac{\partial \bar{w}}{\partial \bar{z}}\right] - \frac{1}{(1+\nu)}(\bar{\varepsilon}^P_{zz} + \Delta\bar{\varepsilon}^P_{zz}) \qquad (3.17)$$

$$\bar{\sigma}_{xy} = \frac{1}{2(1+\nu)}\left[\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right] - \frac{1}{(1+\nu)}(\bar{\varepsilon}^P_{xy} + \Delta\bar{\varepsilon}^P_{xy}) \qquad (3.18)$$

$$\bar{\sigma}_{yz} = \frac{1}{2(1+\nu)}\left[\frac{\partial \bar{v}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{y}}\right] - \frac{1}{(1+\nu)}(\bar{\varepsilon}^P_{yz} + \Delta\bar{\varepsilon}^P_{yz}) \qquad (3.19)$$

$$\bar{\sigma}_{zx} = \frac{1}{2(1+\nu)}\left[\frac{\partial \bar{w}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{z}}\right] - \frac{1}{(1+\nu)}\left(\bar{\varepsilon}^{p}_{zx} + \Delta\bar{\varepsilon}^{p}_{zx}\right) \tag{3.20}$$

and

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + C_1\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right) = C_2 \frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right)$$

$$+ 2C_1\left[\frac{\partial}{\partial \bar{x}}\left(\bar{\varepsilon}^{p}_{xx} + \Delta\bar{\varepsilon}^{p}_{xx}\right) + \frac{\partial}{\partial \bar{y}}\left(\bar{\varepsilon}^{p}_{yx} + \Delta\bar{\varepsilon}^{p}_{yx}\right)\right. \tag{3.21}$$

$$\left. + \frac{\partial}{\partial \bar{z}}\left(\bar{\varepsilon}^{p}_{zx} + \Delta\bar{\varepsilon}^{p}_{zx}\right)\right]$$

$$\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + C_1\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2}\right) = C_2 \frac{\partial}{\partial \bar{y}}\left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right]$$

$$+ 2C_1\left[\frac{\partial}{\partial \bar{x}}\left(\bar{\varepsilon}^{p}_{xy} + \Delta\bar{\varepsilon}^{p}_{xy}\right) + \frac{\partial}{\partial \bar{y}}\left(\bar{\varepsilon}^{p}_{yy} + \Delta\bar{\varepsilon}^{p}_{yy}\right)\right. \tag{3.22}$$

$$\left. + \frac{\partial}{\partial \bar{z}}\left(\bar{\varepsilon}^{p}_{zy} + \Delta\bar{\varepsilon}^{p}_{zy}\right)\right]$$

$$\frac{\partial^2 \bar{w}}{\partial \bar{z}^2} + C_1\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2}\right) = C_2 \frac{\partial}{\partial \bar{z}}\left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}\right]$$

$$+ 2C_1\left[\frac{\partial}{\partial \bar{x}}\left(\bar{\varepsilon}^{p}_{xz} + \Delta\bar{\varepsilon}^{p}_{xz}\right) + \frac{\partial}{\partial \bar{y}}\left(\bar{\varepsilon}^{p}_{yz} + \Delta\bar{\varepsilon}^{p}_{yz}\right)\right. \tag{3.23}$$

$$\left. + \frac{\partial}{\partial \bar{z}}\left(\bar{\varepsilon}^{p}_{zz} + \Delta\bar{\varepsilon}^{p}_{zz}\right)\right]$$

where

$$C_1 = \frac{1 - 2\nu}{2(1 - \nu)}$$

and

$$C_2 = - \frac{1}{2(1 - \nu)}$$

3.2 Solution of Elasto-Plastic Problem

The equations (3.21), (3.22) and (3.23) are compared with the Navier-Cauchy equations (2.5) through (2.7) described in Chapter 2. In the new equations, the field quantities $\bar{x}_i$'s and $\bar{u}_i$'s are used, which are now non-dimensional quantities defined by the equations (3.11) and (3.12). Another difference between these two sets of equations, is the presence of additional terms containing plastic strains. These differences do not alter the solution procedure described in the preceding chapter. The line method is still used to solve the equations (3.21) through (3.23).

The equations (3.18) through (3.20) give the value of shearing stresses. Since the plastic shearing strains are always zero on the free surfaces, except in the case when the shearing traction is prescribed on the boundary, the plastic strain terms can be dropped in these equations for the boundary shearing tractions. The new equations will be in terms of non-dimensional quantities, but the type of

C-2

equations remains the same as used in Chapter 2 to eliminate the imaginary lines at the boundaries. This result offers considerable help in extending the method described in the previous chapter for the elasto-plastic problem. To achieve the complete equivalence we define the non-dimensional spacings $\bar{h}_x$, $\bar{h}_y$ and $\bar{h}_z$ as

$$\begin{aligned} \bar{h}_x &= \frac{h_x}{B} \\ \bar{h}_y &= \frac{h_y}{B} \\ \bar{h}_z &= \frac{h_z}{B} \end{aligned} \tag{3.24}$$

where h_x, h_y and h_z are the line spacings for the x, y, and z sets of lines as shown in Figure 3 and B is the thickness of the cracked specimen.

Ordinary differential equations for the set of x- directional lines are obtained by replacing x_i, u_i and h_i $(i = 1,2,3)$ by the corresponding non-dimensional quantities $\bar{x}_i$, $\bar{u}_i$, $\bar{h}_i$ $(i = 1,2,3)$ in equation (2.22). Using the matrix notations, the new equations for the elasto-plastic case are

$$\underset{\ell\times 1}{\frac{d^2\{\bar{u}\}}{d\bar{x}^2}} = \underset{\ell\times\ell}{[\bar{K}_x]}\underset{\ell\times 1}{\{\bar{u}\}} + \frac{d}{d\bar{x}}\underset{\ell\times 1}{\{\bar{r}(\bar{x})\}} + \frac{d}{d\bar{x}}\underset{\ell\times 1}{\{\bar{r}_1^p(\bar{x})\}} + \underset{\ell\times 1}{\{\bar{r}_2^p(\bar{x})\}} \tag{3.25}$$

Matrix $[\bar{K}_x]$ and vectors $\{\bar{u}\}$, $\{\bar{r}(\bar{x})\}$ are obtained from the equations (2.23) and (2.24) by using the corresponding non-dimensional quantities. The vectors $\{r_1^p(\bar{x})\}$ and $\{r_2^p(x)\}$ are new additions to the equations containing terms of plastic strains. They are considered functions of $\bar{x}$ along the $\bar{x}$- directional lines and expressed as follows,

$$\underset{\ell\times 1}{\{\bar{r}_1^p(\bar{x})\}} = \begin{bmatrix} 2C_1(\bar{\varepsilon}^p_{xx} + \Delta\bar{\varepsilon}^p_{xx})\big|_1 \\ 2C_1(\bar{\varepsilon}^p_{xx} + \Delta\bar{\varepsilon}^p_{xx})\big|_2 \\ \vdots \\ \vdots \\ 2C_1(\bar{\varepsilon}^p_{xx} + \Delta\bar{\varepsilon}^p_{xx})\big|_{\ell-1} \\ 2C_1(\bar{\varepsilon}^p_{xx} + \Delta\bar{\varepsilon}^p_{xx})\big|_{\ell} \end{bmatrix} \tag{3.26}$$

and

$$\underset{\ell\times 1}{\{r_2^p(\bar{x})\}} = \begin{bmatrix} 2C_1[\frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}^p_{yx} + \Delta\bar{\varepsilon}^p_{yx}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^p_{zx} + \Delta\bar{\varepsilon}^p_{zx})]_1 \\ 2C_1[\frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}^p_{yx} + \Delta\bar{\varepsilon}^p_{yx}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^p_{zx} + \Delta\bar{\varepsilon}^p_{zx})]_2 \\ \vdots \\ \vdots \\ 2C_1[\frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}^p_{yx} + \Delta\bar{\varepsilon}^p_{yx}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^p_{zx} + \Delta\bar{\varepsilon}^p_{zx})]_{\ell-1} \\ 2C_1[\frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}^p_{yx} + \Delta\bar{\varepsilon}^p_{yx}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^p_{zx} + \Delta\bar{\varepsilon}^p_{zx})]_{\ell} \end{bmatrix} \tag{3.27}$$

The equation (3.25) is modified by following the procedure described in section (2.3) to obtain a set of first order ordinary differential equations. These equations in matrix form are

$$\frac{d\bar{U}}{d\bar{x}} = \bar{A}_1\bar{U} + \frac{d\ \bar{R}(\bar{x})}{d\bar{x}} + \frac{d\bar{R}_1^p(\bar{x})}{d\bar{x}} + \bar{R}_2^p(\bar{x}) \tag{3.28}$$

The vectors $\bar{R}_1^p(\bar{x})$ and $\bar{R}_2^p(\bar{x})$ are defined as

$$\bar{R}_1^p(\bar{x}) = \begin{bmatrix} \{o\}_{\ell x1} \\ \{\bar{r}_1^p(\bar{x})\}_{\ell x1} \end{bmatrix} \tag{3.29}$$

and

$$\bar{R}_2^p(\bar{x}) = \begin{bmatrix} \{o\}_{\ell x1} \\ \{\bar{r}_2^p(\bar{x})\}_{\ell x1} \end{bmatrix} \tag{3.30}$$

The definitions of other matrices and vectors used in equation (3.28) are the same as given by the equations (2.42) through (2.44) but in this case non-dimensional quantities are employed to formulate the respective matrices.

For the set of y- directional lines. The equation (2.32) is expanded to include the plastic strain terms. Using the matrix notation and the non-dimensional quantities, the new second order ordinary differential equations for the elasto-plastic case are,

$$\underset{m\times 1}{\frac{d^2\{\bar{v}\}}{d\bar{y}^2}} = \underset{m\times m}{[\bar{K}_y]}\underset{m\times m}{\{\bar{v}\}} + \underset{m\times 1}{\frac{d\{\bar{s}(\bar{y})\}}{d\bar{y}}} + \frac{d}{d\bar{y}}\underset{m\times 1}{\{\bar{s}_1^P(\bar{y})\}} + \underset{m\times 1}{\{\bar{s}_2^P(\bar{y})\}} \tag{3.31}$$

The matrix $[\bar{K}_y]$ and vectors $\{\bar{v}\}$, $\{\bar{s}(\bar{y})\}$ are obtained by following a similar procedure used for equations (2.33) and (2.34). The vectors $\{\bar{s}_1^P(\bar{y})\}$ and $\{\bar{s}_2^P(\bar{y})\}$ are new additions containing the terms of plastic strains. They are considered functions of $\bar{y}$ along the y- directional lines and expressed as

$$\underset{m\times 1}{\{\bar{s}_1^P(\bar{y})\}} = \begin{bmatrix} 2C_1(\bar{\varepsilon}^P_{yy} + \Delta\bar{\varepsilon}^P_{yy})|_1 \\ 2C_1(\bar{\varepsilon}^P_{yy} + \Delta\bar{\varepsilon}^P_{yy})|_2 \\ \vdots \\ 2C_1(\bar{\varepsilon}^P_{yy} + \Delta\bar{\varepsilon}^P_{yy})|_{m-1} \\ 2C_1(\bar{\varepsilon}^P_{yy} + \Delta\bar{\varepsilon}^P_{yy})|_m \end{bmatrix} \tag{3.32}$$

and

$$\underset{m\times 1}{\{\bar{s}_2^P(\bar{y})\}} = \begin{bmatrix} 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}^P_{xy} + \Delta\bar{\varepsilon}^P_{xy}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^P_{zy} + \Delta\bar{\varepsilon}^P_{zy})]_2 \\ 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}^P_{xy} + \Delta\bar{\varepsilon}^P_{xy}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^P_{zy} + \Delta\bar{\varepsilon}^P_{zy})] \\ \vdots \\ 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}^P_{xy} + \Delta\bar{\varepsilon}^P_{xy}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^P_{zy} + \Delta\bar{\varepsilon}^P_{zy})]_{m-1} \\ 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}^P_{xy} + \Delta\bar{\varepsilon}^P_{xy}) + \frac{\partial}{\partial\bar{z}}(\bar{\varepsilon}^P_{zy} + \Delta\bar{\varepsilon}^P_{zy})]_m \end{bmatrix} \tag{3.33}$$

A set of first order ordinary differential equations are derived from the equation (3.31) by following the procedure described in section (2.3). Introducing the matrix notation they are written as,

$$\frac{d\bar{V}}{d\bar{y}} = \bar{A}_2\bar{V} + \frac{d\bar{S}(\bar{y})}{d\bar{y}} + \frac{d\bar{S}_1^P(\bar{y})}{d\bar{y}} + \bar{S}_2^P(\bar{y}) \tag{3.34}$$

The vectors $\bar{S}_1^P(\bar{y})$ and $\bar{S}_2^P(\bar{y})$ are defined as

$$\bar{S}_1^P(\bar{y}) = \begin{bmatrix} \{o\}_{m\times 1} \\ \{\bar{s}_1^P(y)\}_{m\times 1} \end{bmatrix} \tag{3.35}$$

and

$$\bar{S}_2^P(\bar{y}) = \begin{bmatrix} \{o\}_{m\times 1} \\ \{\bar{s}_2^P(\bar{y})\}_{m\times 1} \end{bmatrix} \tag{3.36}$$

The definitions of the other matrices and vectors used in equation (3.34) are the same as given by equations (2.46) through (2.48), but in this case non-dimensional quantities are employed to formulate the respective matrices.

Similarly the ordinary differential equations for the set of z-directional lines are obtained by extending the equation (2.38) to include the plastic strain terms. Using the matrix notations, the new equations for the elasto-plastic case are

$$\underset{n\times 1}{\frac{d^2\{\bar{w}\}}{d\bar{z}^2}} = \underset{n\times n}{[\bar{K}_z]}\ \underset{n\times 1}{\{\bar{w}\}} + \frac{d}{d\bar{z}}\ \underset{n\times 1}{\{\bar{t}(\bar{z})\}} + \frac{d}{d\bar{z}}\ \underset{n\times 1}{\{\bar{t}_1^P(\bar{z})\}} + \underset{n\times 1}{\{\bar{t}_2^P(\bar{z})\}} \tag{3.37}$$

The matrix $[\bar{K}_z]$ is formed by using the non-dimensional spacings $\bar{h}_x$ and $\bar{h}_y$ in place of h_x and h_y in various submatrices of matrix $[K_z]$ given by the equation (2.39). The vectors $\{\bar{w}\}$ and $\{\bar{t}(z)\}$ are formed by using the non-dimensional quantities in the equation (2.40). New vectors $\{\bar{t}_1^P(\bar{z})\}$ and $\{\bar{t}_2^P(\bar{z})\}$ are considered functions of $\bar{z}$ along the set of z- directional lines. They are expressed as follows,

$$\underset{n\times 1}{\{\bar{t}_1^P(z)\}} = \begin{bmatrix} 2C_1(\bar{\varepsilon}_{zz}^P + \Delta\bar{\varepsilon}_{zz}^P)|_1 \\ 2C_1(\bar{\varepsilon}_{zz}^P + \Delta\bar{\varepsilon}_{zz}^P)|_2 \\ \vdots \\ 2C_1(\bar{\varepsilon}_{zz}^P + \Delta\bar{\varepsilon}_{zz}^P)|_{n-1} \\ 2C_1(\bar{\varepsilon}_{zz}^P + \Delta\bar{\varepsilon}_{zz}^P)|_n \end{bmatrix} \tag{3.38}$$

and

$$\underset{n\times 1}{\{\bar{t}_2^P(\bar{z})\}} = \begin{bmatrix} 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}_{xz}^P + \Delta\bar{\varepsilon}_{xz}^P) + \frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}_{yz}^P + \Delta\bar{\varepsilon}_{yz}^P)]_1 \\ 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}_{xz}^P + \Delta\bar{\varepsilon}_{xz}^P) + \frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}_{yz}^P + \Delta\bar{\varepsilon}_{yz}^P)]_2 \\ \vdots \\ 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}_{xz}^P + \Delta\bar{\varepsilon}_{xz}^P) + \frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}_{yz}^P + \Delta\bar{\varepsilon}_{yz}^P)]_{n-1} \\ 2C_1[\frac{\partial}{\partial\bar{x}}(\bar{\varepsilon}_{xz}^P + \Delta\bar{\varepsilon}_{xz}^P) + \frac{\partial}{\partial\bar{y}}(\bar{\varepsilon}_{yz}^P + \Delta\bar{\varepsilon}_{yz}^P)]_n \end{bmatrix} \tag{3.39}$$

A set of the first order ordinary differential equations are derived from the equation (3.37) using the procedure described in the section (2.3). The equations in matrix form are,

$$\frac{d\bar{W}}{d\bar{z}} = \bar{A}_3\bar{W} + \frac{d\bar{T}(\bar{z})}{d\bar{z}} + \frac{d\bar{T}_1^p(\bar{z})}{d\bar{z}} + \bar{T}_2^p(\bar{z}) \tag{3.40}$$

The vectors $\bar{T}_1^p(\bar{z})$ and $\bar{T}_2^p(\bar{z})$ are defined as,

$$\bar{T}_1^p(\bar{z}) = \begin{bmatrix} \{o\}_{nx1} \\ \{\bar{t}_1^p(\bar{z})\}_{nx1} \end{bmatrix} \tag{3.41}$$

and

$$\bar{T}_2^p(\bar{z}) = \begin{bmatrix} \{o\}_{nx1} \\ \{\bar{t}_2^p(\bar{z})\}_{nx1} \end{bmatrix} \tag{3.42}$$

The definitions of the other matrices and vectors used in equation (3.40) are the same as given by equation (2.50) through (2.52), but in this case non-dimensional quantities are employed to formulate the respective matrices.

The equations (3.28), (3.34) and (3.40) represent a set of two point boundary value problems, in which half of the boundary conditions are prescribed on one face and the remaining boundary conditions, in general, are given on the second face. They are either in the form of prescribed displacements or normal tractions. For example, for the set of x- directional lines, we have prescribed surface tractions on the face IV and face I, as shown in Figure 4. To obtain the starting

vectors for the recurrence relations method, the stress displacement relations given by equations (3.15) through (3.20) are used, while the procedure essentially remains the same as described in section 2.3.2. The augmented recurrence relations given by equation (2.91) are used.

The equations (3.28), (3.34) and (3.40) are coupled differential equations. The coupling terms on the right hand side involve $\bar{u}$, $\bar{v}$, $\bar{w}$, their derivatives and plastic strains $\bar{\varepsilon}^p_{ij}$ $(i,j = 1,2,3)$. The solution is obtained in an iterative manner and the same iteration is used to evaluate both the displacement terms and the plastic strain terms. The solution of these equations yield a field solution for the displacements $\bar{u}_i$'s and the total strains $\bar{\varepsilon}_{xx}$, $\bar{\varepsilon}_{yy}$ and $\bar{\varepsilon}_{zz}$. The total shearing strains are obtained by using the equation

$$\bar{\varepsilon}_{\substack{ij \\ i\neq j}} = \frac{1}{2} (\bar{u}_{i,j} + \bar{u}_{j,i}) \qquad i,j = 1,2,3 \tag{3.43}$$

The actual evaluation will involve the use of three-point finite difference equations. To calculate the plastic strains we make use of the following equations (26).

$$\Delta\bar{\varepsilon}^p_x = \frac{\Delta\bar{\varepsilon}_p}{3\bar{E}_{et}} (2\,\bar{\varepsilon}'_x - \bar{\varepsilon}'_y - \bar{\varepsilon}'_z)$$

$$\Delta\bar{\varepsilon}^p_y = \frac{\Delta\bar{\varepsilon}_p}{3\bar{\varepsilon}_{et}} (2\bar{\varepsilon}'_y - \bar{\varepsilon}'_z - \bar{\varepsilon}'_x)$$

$$\Delta\bar{\varepsilon}_z = \frac{\Delta\bar{\varepsilon}_p}{3\bar{E}_{et}} (2\bar{\varepsilon}'_z - \bar{\varepsilon}'_x - \bar{\varepsilon}'_y)$$

$$\Delta\bar{\varepsilon}^p_{xy} = \frac{\Delta\bar{\varepsilon}_p}{\bar{\varepsilon}_{et}} \bar{\varepsilon}'_{xy}$$

$$\Delta\bar{\varepsilon}^p_{yz} = \frac{\Delta\bar{\varepsilon}_p}{\bar{\varepsilon}_{et}} \bar{\varepsilon}'_{yz}$$

$$\Delta\bar{\varepsilon}^p_{zx} = \frac{\Delta\bar{\varepsilon}_p}{\bar{\varepsilon}_{et}} \bar{\varepsilon}'_{zx} \tag{3.44}$$

where
$$\bar{\varepsilon}'_{ij} = \bar{\varepsilon}_{ij} - \sum_{\ell=1}^{k-1} \bar{\varepsilon}^p_{ij} \qquad i,j = 1,2,3 \tag{3.45}$$

k refers to the current increment number.

$$\bar{\varepsilon}_{et} = \frac{\sqrt{2}}{3} [(\bar{\varepsilon}'_x - \bar{\varepsilon}'_y)^2 + (\bar{\varepsilon}'_y - \bar{\varepsilon}'_z)^2 + (\bar{\varepsilon}'_z - \bar{\varepsilon}'_x)^2$$

$$+ 6(\bar{\varepsilon}'_{xy})^2 + 6(\bar{\varepsilon}'_{yz})^2 + 6(\bar{\varepsilon}'_{zx})^2]^{\frac{1}{2}} \tag{3.46}$$

and
$$\Delta\bar{\varepsilon}_p = \frac{\bar{\varepsilon}_{et} - \frac{2}{3}\frac{(1+\nu)}{E}\bar{\sigma}_{e,k-1}}{1 + \frac{2}{3}(\frac{1+\nu}{E})(\frac{d\sigma_e}{d\varepsilon_p})_{k-1}} \tag{3.47}$$

The term $\frac{1}{E}(\frac{d\sigma_e}{d\varepsilon_p})_{k-1}$ is determined from the stress-strain diagram of the material. The non-dimensional effective stress $\bar{\sigma}_e$ is defined as

$$\bar{\sigma}_e = \frac{1}{\sqrt{2}}[(\bar{\sigma}_x - \bar{\sigma}_y)^2 + (\bar{\sigma}_y - \bar{\sigma}_z)^2 + (\bar{\sigma}_z - \bar{\sigma}_x)^2$$

$$+ 6(\bar{\sigma}^2_{xy} + \bar{\sigma}^2_{yz} + \bar{\sigma}^2_{zx})]^{\frac{1}{2}} \tag{3.48}$$

The method, applied to the present problem is known as the method of successive elastic solutions (26). At first an elastic solution is obtained. It is scaled up such that the highest stressed node point in the grid reaches the incipient yield condition. For the subsequent loading the loading path is divided into a number of sufficiently small increments. The iterative procedure for determining the incremental plastic strains for each load increment is as follows,

1) The applied load is calculated by adding the loading increment to the current load level.
2) Initial values of plastic strain increments are all assumed to be zero in the beginning of the load increment.
3) Obtain the field solution for the total strains $\bar{\varepsilon}_{ij}(i,j = 1,2,3)$ by solving the equations (3.28), (3.34) and (3.40).
4) Calculate the modified total strains from equation (3.45) and evaluate the equivalent modified total strain $\bar{\varepsilon}_{et}$ from equation (3.46).
5) Find the equivalent plastic strain increment $\Delta\bar{\varepsilon}_p$ from equation (3.47) in which $\bar{\sigma}_{e,k-1}$ is the dimensionless value of the equivalent stress at the end of the previous increment of loading. For the first load increment and also for the case where there was no plastic flow under previous loading, $\bar{\sigma}_{e,k-1}$ is equal to the dimensionless yield stress $\bar{\sigma}_o$, i.e., unity.
6) Calculate new set of incremental plastic strains from equations (3.44).

7) Check for the convergence of the incremental plastic strains.

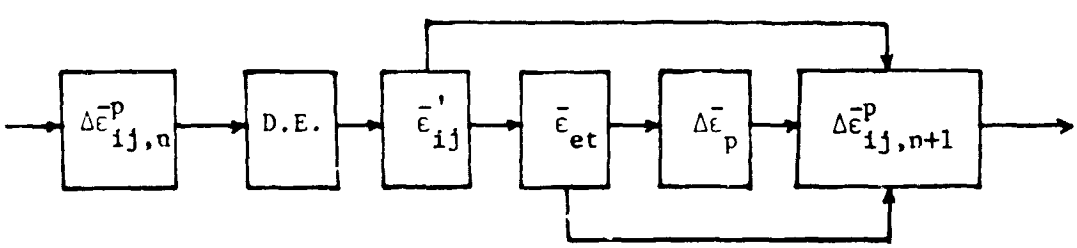
The convergence criterion used is

$$\frac{\sum_{m=1}^{N} \left[\left| \frac{\bar{\varepsilon}^{n+1}_{yy,m} - \bar{\varepsilon}^{n}_{yy,m}}{\bar{\varepsilon}^{n+1}_{yy,m}} \right| \right]}{N} < EPS \tag{3.49}$$

N represents the total number of yielded nodes, while n is iteration number and m is a grid point identifier. The value of the EPS is chosen depending upon the desired accuracy of the solution. Repeat steps 3 to 6 until the convergence is achieved on the incremental plastic strains.

8) Sum the plastic strain increments and return to step 1.

Once the successive approximation procedure has converged, the stresses are calculated at all the grid points. The schematic representation of the method is given below,



CHAPTER 4

STRESS INTENSITY FACTORS AND J-INTEGRAL DETERMINATION

The stress intensity factor (SIF) and the path independent J-integral proposed by Rice (27) are most commonly used fracture mechanics parameters. Considerable effort (28,29,30) has been devoted to develop the techniques for an accurate determination of these parameters.

In the next section a commonly used technique to evaluate the stress intensity factors is discussed. Finally, a procedure is given to determine the path independent J-integral.

4.1 Determination of Stress Intensity Factors

The stress intensity factor (SIF) is defined as

$$K_I = \lim_{R \to 0} \sigma_y \, (2\pi R)^n \tag{4.1}$$

where σ_y is the stress ahead of crack tip, R is the distance measured from the crack tip. n is the singularity of the stress field in the neighborhood of the crack. It was found, however, that due to the coarseness of the grid used, the usual extrapolation techniques did not yield accurate results. The precise crack tip location is also unknown, except that it is approximately midway between two lines. To overcome these problems, a procedure described in reference (22) is followed. In this method two terms in the displacement series expansions around the crack tip are retained rather than one. This also permits to determine the actual crack tip location from the computed results.

$$v|_{y=0} = \alpha K_I [\sqrt{\frac{R + r}{2\pi}} + \frac{L_I}{K_I} \sqrt{(R + r)^3}] \qquad (4.2)$$

where α is a function of Poisson's ratio, and r is the crack edge position correction, measured from the originally assumed mid point position. Using displacement data from the adjacent nodes to the crack edge in equation (4.2), values of the αK_I, $\frac{L_I}{K_I}$ and r are calculated for each value of z. The distance R is measured from the halfway point between nodes specifying boundary stresses and displacements, respectively. A plane strain condition is assumed to exist all through the thickness of the specimen, except at the surface. Correspondingly α would be equal to 3.56 for the plane strain case and 4.0 for the plane stress case.

In the case of the curved crack fronts, the v- displacement values are extrapolated along the normal directions to the crack front, with the help of nodal displacements. These values are used in equation (4.2) to evaluate the local stress intensity factors through the thickness of a cracked specimen.

4.2 J-integral Determination

Rice (27) developed a path independent integral, known as the J-integral in the literature. This integral is associated with the change in energy of a body due to the crack growth and it is expressed as

$$J = \int_{\Gamma} (W(\varepsilon)\ dy - T_i \frac{\partial u_i}{\partial x}\ ds) \qquad (4.3)$$

Here Γ is a curve surrounding the notch tip as shown in Figure 9(a). The contour integral is evaluated in a counter clockwise sense, starting from the crack surface. The strain energy density $W(\varepsilon)$ is defined as

$$W(\varepsilon_{mn}) = \int_0^{\varepsilon_{mn}} \sigma_{ij} \, d \, \varepsilon_{ij} \tag{4.4}$$

and the traction vector T_i is

$$T_i = \sigma_{ij} \, n_j \tag{4.5}$$

u_i is the displacement vector.

The consideration of singular terms associated with the stress environment near the crack tip in a linearly elastic body, in the evaluation of equation (4.3) leads to well known relationships as obtained by Rice (27). These relations are,

$$\begin{aligned} J &= \frac{1 - \nu^2}{E} K_I^2 \qquad \text{(for plane strain)} \\ J &= \frac{1}{E} K_I^2 \qquad \text{(for plane stress)} \end{aligned} \tag{4.6}$$

The equations (4.6) could be used to determine stress intensity factors by computing J-values which can be obtained without a detailed knowledge of the stress and strain field, very near the crack tip.

By taking advantage of geometric and loading symmetry about the x axis, retangular paths are chosen to calculate J- integral values. For a rectangular path shown in Figure 9(b) the contour integral for J can be written as,

$$J = 2\int_1^2 [W(\varepsilon) - \sigma_x\,\varepsilon_x - \sigma_{xy}\,\frac{\partial v}{\partial x}]dy$$

$$+ 2\int_2^3 (\sigma_{xy}\,\varepsilon_x + \sigma_y\,\frac{\partial v}{\partial x})\,dx$$

$$+ 2\int_3^4 W(\varepsilon) - \sigma_x\,\varepsilon_x - \sigma_{xy}\,\frac{\partial v}{\partial x}]dy \qquad (4.7)$$

where

$$W(\varepsilon) = \int_0^{\varepsilon_{mn}} (\sigma_x\,d\varepsilon_x + \sigma_y\,d\varepsilon_y + \sigma_z\,d\varepsilon_z$$

$$+ 2\sigma_{xy}\,d\varepsilon_{xy} + 2\sigma_{yz}\,d\varepsilon_{yz} + 2\sigma_{zx}\,d\varepsilon_{zx}) \qquad (4.8)$$

The coordinates of the points shown in Figure 9(b) are used as integration limits in equation (4.7) and the integrations are calculated using the trapezoidal formula of numerical integration.

CHAPTER 5

RESULTS AND DISCUSSION

A computer program was written to solve the elastic problem for the different cracked specimens. This program was later modified to include the plasticity terms. Since the ordinary differential equations are coupled, a successive approximation procedure was used to obtain the solution. The computation for all the examples were performed on IBM 370/3033 time sharing computer.

5.1 Single Edge-Notched Tensile Specimen

The new formulation of the MOL was first applied to a single edge notched specimen in tensile loading. The specimen shown in Figure 2 has dimensions, $W/a = 2$, $B/a = 3$, $L/a = 3.5$, where a, B, W and L are the specimen crack length, thickness, width and length respectively.

Figure 10 shows a plot of the dimensionless maximum crack opening versus the number of grid lines. We note the following:

1) This displacement approaches smoothly an asymptotic value. 2) The value obtained in reference (31) using the old formulation of MOL, is in error by approximately 17% whereas the error using the same number of lines with the new formulation is only about 3 percent from the asymptotic value. Furthermore, 12 x- grid lines were the most that could be used in reference (31), before the solution diverged. 3) A comparison with the results obtained in reference (32) using finite elements, indicates that the present solution using 16-x lines, is essentially correct since it was indicated in reference (32) that the value obtained there may be as much as 25 percent low.

The dimensionless stress intensity factor for this specimen at the mid thickness is shown in Figure 11 versus the number of x grid lines. Again, the value obtained in reference (31) using the old formulation is in error by about 8 percent, whereas the new formulation with the same number of grid lines is in error by about 3 percent from the asymptotic value.

Figure 12 shows the stress distribution ahead of the crack tip at the mid thickness for different grids. Note that the same distribution is obtained with 12, 14 or 16 x grid lines. The larger the number of x grid lines, however, the closer the crack tip is approached, and the more accurate is the stress intensity factor calculation.

The variation of the stress intensity factor through the thickness of the specimen is shown in Figure 13. The SIF decreases approximately 15 percent in going from the center of the specimen to the surface.

5.2 Compact Tension Specimen

The standard compact tension specimen (CTS) shown in Figure 1(a) is considered next. The dimensions of this specimen are $W/a = 2$, $B/a = 1$, $L/a = 2.4$. The load was assumed to be parabolic shear load along the pin load line as shown in Figure 1(b).

Figure 14 shows the dimensionless crack opening displacements for this specimen. As is seen, there is very little difference between the midplane and surface displacements. The experimental value from reference (25) is also shown and it is seen that good agreement is obtained.

The variation of stress intensity factor through the thickness of the compact tension specimen is shown in Figure 15. Also, shown in this figure are some of the results obtained by finite elements in

reference (11), which treats the Battelle Benchmark geometris, (33). Several different methods were investigated in reference (11) for evaluating the stress intensity factors from the finite element results. These were displacement substitution, modified displacement substitution, virtual crack extension and contour integration methods, designated as methods D, E, V, and J respectively in reference (11).

In the displacement substitution method, the displacement at the vertex and mid-edge nodes in the special elements on the face of the crack were used. Since a coarse mesh was used in reference (11), an extrapolation was done on the values of SIF obtained by displacement substitution from the vertex and mid-edge nodes and the method is named as modified displacement method.

In the virtual crack extension method the evaluation of the energy change is made corresponding to a small adjustment in the position of any point on the tip of the crack in any direction. As shown by Rice (27), the component J_x equals the rate of change of energy per unit area of crack extension at a point in the direction of the x- axis. The path independent J- integral proposed by Rice (27) is used in the method J.

The results of methods D, E and V have been plotted in Figure 15 together with the results obtained herein using the MOL. The results obtained from MOL lies approximately midway between the methods V and E. The result from the two-dimensional plane strain solution given in ASTM E399-78(5) is also shown in Figure 15. The MOL result is only 0.6 percent higher than this result at the center where plane strain is

expected to exist. The two dimensional solution is obtained by boundary collocation method and believed to have an accuracy of 0.2 percent for a CT specimen. Note also the drop of approximately 15 percent in SIF value as one moves from center to the surface of the specimen.

From the above results it is apparent that the line method of analysis as modified herein appreciably increases the accuracy and reduces convergence problems.

5.3 Curved Crack Front Specimens

The method of lines was next applied to curved crack front (CCF) problems. The compact tension specimen in tensile loading is considered first. An accurate geometrical description of a curved crack front is not possible if a coarse grid such such as NX = 12, NY = 6, NZ = 5 is used. Therefore, two fine grids of NX = 20, NY = 7, NZ = 5 lines and NX = 20, NY = 7, NZ = 7 lines were used in the subsequent analyses. Since approximate parabolic curves were used in the study it was difficult to assign a single measurement for the crack front curvature. To describe this curvature, therefore a parameter called crack tunnel depth is defined as the difference between the crack lengths at the center and the surface of the specimen. Thus an increasing crack tunnel depth reflects an increasing crack curvature.

Figure 16 shows a plot of the local stress intensity factor for a CT specimen in tensile loading. Curve 1 shows the SIF variation with thickness for a straight crack front. For this case maximum SIF occurs at the center of the specimen and it drops by 13 percent on the surface. This trend is in accordance with the previously reported results of Raju (10). For the crack tunnel depth of 0.1 it is seen

that SIF decreases at the center by 9.5 percent, while at the surface its value increases by 20.5 percent. Similar type of variation is reported by Pereira et. al. (14). However, a direct comparison of the results is not possible because they used a different crack and specimen size. For higher crack tunnel depths the maximum SIF value no longer occurs at the surface. It shifts to the adjacent interior point of the grid. This trend is not reported in reference (14), probably because a crack length (a/W) of 0.25 was used which is 1/2 of the one used in the present study and more importantly three layers of elements were used in the thickness direction, making it probable that the maximum SIF value was missed.

Figure 17 shows the variation of non-dimensional SIF for the CT specimen with assumed parabolic loading along the crack line instead of tensile loading. The dimensions of the specimen are the same as before. Once again for the crack tunnel depth of 0.097 the local SIF value decreases at the center while its value rises at the surface of the specimen. For the higher crack tunnel depths the trend of lowering the SIF value at the center of the specimen continues. The maximum SIF no longer occurs at the surface, but it shifts in the interior of the specimen. This is similar to the observed trend for the tensile loading.

A possible explanation for these trends could be given by considering the various effects which have direct bearing on the SIF values. The rise in the SIF value at the surface is mainly due to the presence of a crack front with curvature. However, there are two other effects

which are responsible for the reduction in the SIF value at the surface. Firstly, the existence of plane stress condition at the surface leads to a drop in SIF value (an approximate drop of 15 percent was observed for a straight crack front). Secondly, due to the presence of a curved crack front, the crack length decreases at the surface as compared to the center, leading to a reduction in the SIF value. As it could be seen that for the smaller crack tunnel depth the effect of curvature is dominant over other effects and thus the maximum SIF value occurs at the surface. However, for the curved crack fronts with larger crack tunnel depths, the effect of reduction in the crack length seems to be dominating. Consequently the SIF value at the surface no longer remains maximum. A combination of the three effects, mentioned earlier, determines the location of maximum SIF value through the thickness of the CT specimen.

Figure 18 shows the variation of center, surface and thickness average stress intensity factors with increasing crack tunnel depth for a CT specimen. The stress intensity factors are non-dimensionalized with the respective values of SIF occuring for a straight crack. The thickness average stress intensity factor (K_{av}) is calculated by taking average of the SIF values across the thickness. It is reflective of the average stress environment existing near the crack tip. Note that non-dimensional SIF value at the center of the specimen keeps on decreasing as the crack tunnel depth increases. However, the value of SIF at the surface increases only up to a certain crack tunnel depth. The thickness average SIF also decreases with increasing crack tunnel

depth. Similar behavior for the thickness average SIF was observed by McGowan (12) in his study on a single edge notch specimen. This clearly shows that the presence of curved crack front can significantly change the value of SIF. The lowering in the value of SIF indicates a reduction in stress concentration near the crack tip. Consequently, the specimen can withstand higher loads before showing any appreciable crack growth.

In Figure 19 the change in the thickness average SIF is plotted against a non-dimensional curvature parameter η. The value of η corresponds to a difference between the average crack length (a_{av}) and the surface trace of the crack divided by average crack length (a_{av}). The calculation of a_{av} is done by computing an average of the three crack measurements taken at the center of the crack front and midway between the center and the end of the crack front on each side. This definition of a_{av} is in accordance with the ASTM standard E399-78 (5). The standard restricts the length of either surface trace of the crack to more than 90 percent of the average crack length (a_{av}). Based on this criterion, the ASTM region is marked on the Figure 19. Any test in which the specimen shows a surface trace within the marked region, will be considered a valid test. It could be seen that for the case when there is a surface trace of 90 percent of a_{av} in the specimen, there is a 7.0 percent lowering in the thickness average stress intensity factor. This shows that the use of plane strain formula given in references (5) will result in an overestimation of the fracture toughness of a material by 7.0 percent.

On the basis of experimental results and three-dimensional analysis of Pereira et al. and Neale (14,15), recently, an amendment has been

incorporated (40) to extend the limit on the surface trace of the crack to 85 percent of the average crack length. This corresponds to a η value of 15 percent. A drop of 12.5 percent in the value of thickness average SIF can be observed from the Figure 19 for this value of η. This is an increase of 5.5 percent over the previous value and implies that the use of two-dimensional plane strain formula will result in an overestimation of the fracture toughness by 12.5 percent. These figures provide a fairly good estimate of overestimation of the fracture toughness if the CCF is present in the test specimen.

5.4 Effect of Plastic Flow

The increase in SIF values at the surface of the CT specimens with different curved crack fronts indicates that the crack initiation will start from the surface of the specimen. This is contradictory to the experimentally observed behavior of crack tunnelling. It has been suggested that certain amount of plastic flow takes place at the surface which results in lowering of the SIF values there. To verify this an elasto-plastic stress analysis was carried out for the CT specimen.

The material considered in this study was Aℓ-5083. A stress-strain diagram for this material is shown in Figure 20. The yield stress (σ_o) is taken as 12.417×10^3 psi and the value of elasticity modulus (E) is estimated from the test data provided in reference (34). The plasticity solution was obtained by following an incremental solution procedure. The total numer of increments needed for complete elasto-plastic analysis were between 20 and 26. The incremental load $\frac{\Delta\tau}{\sigma_o}$ was taken approximately equal to 0.075. For one increment, the complete elasto-plastic solution took approximately 6 minutes of CPU time on the IBM 370-3033 computer

and for the convergence criterion described in Chapter 3, a fixed value of 0.001 was taken for EPS.

In Figure 21 the variation of non-dimensional crack opening displacement along the width of the CT specimen is shown. These displacements are extrapolated to determine the crack mouth opening displacements of the specimen. The displacements considered herein are at the center of the specimen for a straight crack front. The different curves represent increasing applied load levels.

An experiment was performed to obtain the load versus crack mouth opening displacement (CMOD) for the compact tension specimen at NASA Lewis Research Center. The specimen was made according to the ASTM Standard E399-78 (5). It was loaded in an MTS automated testing system equipped with a load cell capacity of 20 Kips. The loading was stroke controlled having a rate of 0.02 inch/minute. The main feature of the testing system included a closed-loop, servo controlled hydraulic operation. It was fully computer operated. A clip guage satisfying the requirements of the ASTM E399 standard was mounted on the specimen to measure the CMOD.

The stroke controlled software developed at Lewis Research Center was used for the data acquisition. The load and the corresponding CMOD constitute a data point. These data points were stored on hard disc with an acquisition rate of 1000 data points in half a second. A real time plot of load versus CMOD was drawn on a flat bed plotter.

The load versus CMOD results of the experiment were also printed in a tabular form on a line printer. This numerical data is utilized to draw the experimental curve on Figure 22. Also shown on this figure

is the results obtained from the MOL. At incipient load level where plastic flow begins, the difference between the experimental displacement and the calculated value is 5 percent. There seems to be a fairly good correlation between the results obtained by the experiment and the calculated value.

A complete elasto-plastic analysis was carried out for three different crack fronts. These crack fronts are shown in Figure 23. The first crack front (CCF0) is a straight crack while the other two (CCF1, CCF2) have some curvature. The growth of the plastic zone with load for a straight crack front is shown in Figure 24. The growth is shown at three different locations through the thickness of the specimen, at the center, at the surface and some plane adjacent to the surface. Location adjacent to the surface was chosen because of maximum plastic strain ($\bar{\varepsilon}^p_{yy}$) there. This growth of plastic zone is different from the conventional two-dimensional plane model. Note also the growth of the plastic zone at the back surface of the specimen. In this zone the plastic strains ε^p_{yy} are compressive due to the bending component of the load. For the two other crack fronts CCF1 and CCF2 the development of the plastic zone growth is shown in Figures 25 and 26.

To compare the plastic zones for different crack fronts, the highest load level of 2.5 kips was chosen. At this load level the plastic zones at the three different locations in the thickness of the specimen are plotted in Figure 27. At the center, the plastic zone sizes are approximately the same for CCF0 and CCF1, while it is considerably smaller for CCF2. On the surface the plastic zone is bigger for the CCF2. Note also the presence of a small elastic region within the plastic zone.

This large growth of the plastic zone is anticipated for CCF2, due to the rise in stresses at the surface of the specimen. The plastic zone growth is smaller for the CCFO at the surface as compared to CCF1.

The stresses are relieved due to the large plastic flow at the surface, when the curved crack fronts are present. This explaines the beginning of crack advancement at the center of the specimen. The presence of triaxial constraint due to the stress σ_{zz} at the center also contributes to the tunnelling behavior of the crack.

5.5 Calculation of J-Integral

The J-integral proposed by Rice (27) plays an important role in the non-linear fracture mechanics. It was shown that for linear and non-linear elastic materials, this integral is path independent. Furthermore, based on energy considerations, it was proven to be equivalent to energy release rate per unit crack extension. This conclusion is very important because it removes the need of accurately determining the stresses near the crack tip. J is also used in predicting the elastic-plastic crack growth (35). In the present study J values are calculated for different curved crack fronts to analyze the behavior of crack growth in the compact tension specimen.

Nine different paths were chosen to evaluate the J values at different thicknesses of the CT specimen. These paths are shown in Figure 28. The non-dimensional J values at three different locations are plotted in Figures 29,30 and 31. These results are for a straight crack front at increasing load levels. The following observations are made. 1) For the loads which are close to the load for incipient plastic flow, the J values remain nearly the same for different paths.

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2) For the higher load levels some perturbation is observed in the J-values along the different paths. Similar trends are reported in reference 36 in which a two-dimensional elastic-plastic analysis of a CT specimen is carried out. 3) As we move from center to the surface of the specimen the path independence property of the J-integral deteriorates.

The use of two-dimensional definition of the J-integral in a three-dimensional analysis is debatable but as reported in reference (37) that due to the symmetry of the specimen it is enough to consider only one component J_x of the general three-dimensional vector $\vec{J}$. An equation to evaluate J_x was given, which is similar to the two-dimensional definition of the J-integral with the addition of a surface integral term. The details of the equation can be found in Appendix D. It was further shown in the same study, that the surface integral does not significantly change the numerical value of J. Only at the surface does its evaluation have some effect on the J-integral values. Based on this, in the present study the two dimensional definition of J is adopted.

Figure 32 shows the different paths used to evaluate J-integral values for the curved crack front CCF1. All the paths are the same as used before with the exception of the paths chosen at the surface. Since the crack length is smaller at the surface due to the curvature, all the paths are shifted toward the left. Once again the J values are plotted for different locations through the thickness of the specimen in Figures 33-35. It is observed that for lower load levels the J-integral is essentially path independent but for the higher loads this property does not hold good.

In Figure 36 different paths, used for the curved crack front 2 are shown. Note the shifting of paths towards the left at locations $z = 0.375$ and $z = 0.5$ due to the presence of the curved crack front. The J-integral values are plotted for different paths and locations in Figure 37-39 for CCF2. Once again, it is observed that path independent property of J-integral is closely followed only for the loads which are closer to the load at which plastic flow begins and at the center of the specimen.

From the above results it is clear that J-integral has significant path dependence immediately adjacent to a crack tip under small-scale yielding conditions in an elastic-plastic material. Parks (38) and McMeeking (39) have also reported similar results. This may be due to the unloading which occurs at the nodes near the crack tip because of the plastic flow. These observations very seriously challenge the role of J as a parameter characterizing the crack-tip stress field , at least for materials modelled by the Von Mises flow theory.

In view of these observations it was difficult to choose a representative J-integral value for each of the loading cases. A reasonable choice for J can be obtained by considering Figure 40. The non-dimensional values of the stress intensity factor were calculated using the J values obtained from the path 1. Similar calculations are made for the SIF using the average $\bar{J}$ values which are obtained by averaging all the J values for paths 3 through 9. These numerical values of SIF and the SIF values obtained from crack opening displacements are plotted in Figure 40. It can be seen that the SIF values obtained from the J-integral

values for the path 1 are very close to the SIF values from crack opening displacement. While the average $\bar{J}$ values do not yield an accurate result as one proceeds towards the surface of the specimen. Because of this reason the results from the path 1 for the J-integral are accepted as good estimates for further analysis.

The non-dimensional J-integral values as a function of load, obtained from the use of path 1 are shown in Figures 41-43. Curves for the three different crack fronts are shown in the figures. From figure 41, it could be seen that at the center of the specimen, the J values for CCF1 and CCF2 are lower than the values for CCF0, but at a load of 2.3 kips the J values of CCF0 and CCF1 coincide while for CCF2, it continue to remain lower. On the surface, the J values remain higher for CCF1 and CCF2 for the smaller load levels as shown in Figure 43. But, as load increases this no longer remains true and the J values go down considerably for CCF1. The J values for CCF2 also fall below the J values of CCF0. This shows that due to the plastic flow, J-integral value decreases at the surface for the curved crack fronts as compared to straight crack front.

The variation of J obtained from path 1 is plotted through the thickness on Figures 44-47. For the straight crack front the J values for the lower loads are maximum at the center and minimum at the surface as shown in Figure 44. With the increase in load this trend is slightly modified. The J-integral value at the center still remains the maximum but the minimum value shifts from the surface to some interior point. For CCF1 and CCF2, the J value is maximum at the surface for the loads close to load at which plastic flow begins. For higher loads this trend

reverses for CCF1. The maximum value of J shifts to the center of the specimen while for CCF2 the maximum values do not occur at the center but shifts to an interior point.

The existence of a critical J* value is assumed at which crack advancement starts. From Figure 45, it is obvious that for CCF1 after some plastic flow, the crack advancement will start at the center of the specimen, not at the surface as predicted by a purely elastic analysis. Similar behavior was deduced from the approximate elastic-plastic model, developed by Neale (16) for a CT specimen. This is in full accordance with the crack tunnelling behavior observed during the experiment. For CCF2 the maximum J value will occur at the center of the specimen, as the shift in maximum J value towards the center could be seen in Figure 46 with the increase in load.

CHAPTER 6

SUMMARY AND CONCLUSIONS

An improved formulation of the method of lines (MOL) is presented. The five point finite difference formulas are introduced to achieve more accurate results. The resulting ordinary differential equations are solved by a recurrence relation method. It is a well suited method for solving two point boundary value problems. Two specific geometries namely the edge notch specimen and the compact tension specimen are considered. For both the geometries, complete field solution for the stresses and strains were obtained.

To establish the convergence characteristics of the newly improved MOL, the field solutions were obtained for edge notch specimen for different grid sizes. On comparing the maximum crack opening and the stress intensity factors for different grid sizes, it was found that the solution converges to an asymptotic value. Even the coarse grid such as NX = 12, NY = 6, NZ = 5, yielded fairly good results and the CPU time was only of the order of 2.5 minutes.

For the compact tension specimen maximum crack opening displacement was compared with experimental results and were found to be in good agreement. The SIF value at the center of the compact tension specimen was only 0.6% higher from the value given in the ASTM standard (reference (5)).

Only smaller grid sizes such as NX = 12, NY = 8, NZ = 8 could be used in the old method of lines. For the bigger grid sizes the solution became instable. This places a serious restriction on the shape of the

crack fronts which could be used to study the effect of crack front curvature on the local stress intensity factors. By introducing the new modifications in the MOL, it became possible to use bigger grid sizes such as NX = 20, NY = 7, NZ = 7. Presently, only the core size and the CPU time seem to limit the size of the grid.

The complete solutions were obtained for the compact tension specimen in tensile and shear loading, containing different crack fronts. For the purely elastic case, it was found that as the crack front curvature increases the SIF value at the center of the specimen decreases while increasing at the surface. For higher values of crack front curvatures the maximum value of the SIF occur at an interior point located adjacent to the surface. These results indicate that for the specimen containing a curved crack front, the crack growth will initiate at the surface of the specimen. This conclusion is in direct contradiction with the experimental observations in which a tunnelling behavior is observed, but can be explained by the presence of plastic flow.

A thickness average SIF value was computed. It is assumed that it reflects the average stress environment near the crack edge. On the basis of this, it was estimated that use of the ASTM formula (reference (5)) will lead to an overestimation of the fracture toughness by 7 percent if the curved crack front present in the compact tension specimen just satisfies the ASTM limit on the surface trace of the crack, providing no plastic flow occurs. It was further estimated that the proposed amendment in the ASTM standard on the surface trace of a

crack will lead to a maximum overestimation of the fracture toughness by 12.5 percent.

To investigate the effect of plastic flow in the compact tension specimen, the equations of the method of lines are augmented to include the plasticity terms. Complete elasto-plastic analysis were carried out for three different cases of crack fronts, which include the case of straight crack front.

To check the accuracy of the elasto-plastic analysis, the load versus crack mouth opening displacements were compared with experimental results. The experiment was conducted at NASA, Lewis Research Center. The two results were found in good agreement.

The growth of the plastic zones are compared for a straight crack and two curved crack fronts. The maximum plastic zone occurs at the surface for the curved crack front with the maximum curvature. This qualitatively explains the reason for the initiation of crack growth at the center of the specimen. Due to the large plastic flow at the surface, the stresses are relieved.

To further investigate this, the J-integral values are computed at different locations for each of the crack front. In general, it is observed that the J values are path independent for the lower loads. As the extent of plasticity increases, this property of path independence breaks down. It was also found that as we move from the interior to the surface, this property deteriorates. In view of these results, one specific path was chosen to compare different J values. The choice of the path was based on a comparison between the SIF values obtained from the J-integral values and the SIF values computed using the crack

opening displacements. For this path it was observed that at the load levels close to the elastic conditions the J values are higher at the surface and lower at the center for the curved crack fronts. As the load increases, the trend reverses itself and for the curved crack front with small curvature, the maximum of J value occurs at the center. This provides a quantitative explanation of crack initiation at the center of the specimen. This conclusion can not be drawn from a purely elastic analysis.

6.1 Concluding Remarks

The new improvements in the method of lines, have considerably enhanced the accuracy and the stability of the method. Converged results can be obtained by using ralatively coarse grids. It was observed that increase in the number of lines in one direction only, can lead to an instable solution. In some cases this instability could be removed by adjusting the spacings, used for the application of the recurrence relation method. These spacings were selected in such a way, that they were approximately equal to each other in all the three directions. This was established purely on the basis of numerical experimentation. At this stage, a rigorous mathematical analysis of error estimates and stability of the method is desirable.

In the present work, one particular constant grid size, was chosen for each direction. This use of the constant grid size to approach crack tip as close as possible, leads to a large number of equations to be solved. In any future work this problem can be alleviated by using a varying grid size.

The present formulation of the elastic-plastic problem is based on the successive elastic solution method. It is observed that rate of the convergence slows down as the extent of plasticity increases. The solution procedure start to diverge when the plastic zone starts growing in the compression zone of the compact tension specimen. This shows that present formulation cannot predict large non-linear effects. It will need some modifications to obtain solution for large plasticity effects.

The iterative scheme of the method of lines was based on the successive approximation procedure. This scheme is simplier to adopt, but needs more iterations. To cut down on the iterations, for the elastic case a successive over-relaxation (SOR) parameter was tried. An improved guessed solution was obtained by combining the solution for the current and previous iteration with the help of a SOR parameter. This improved guessed solution was used to carry out the next iteration. In the case of a grid, NX = 20, NY = 7, NZ = 5, the number of iterations were reduced from forty-seven to twenty-eight, when a SOR parameter of value two was used. This leads to a considerable saving in the computer time. Such parameters could be studied in more detail to accelerate the convergence rate of the present method of line.

LIST OF REFERENCES

1. Westergaard, H. M. "Bearing Pressure Cracks", J. Applied Mechanics, 6(2):A49-A53, 1939.

2. Kolosov, V. G. "On an Application of Complex Variable Function Theory to a Plane Problem", In the Mathematical Theory of Elasticity, Yar Yes, 1909.

3. Gross, B., Roberts, E., Jr. and Srawley, J. E. "Elastic Displacements for Various Edge Cracked Plate Specimens", The International Journal of Fracture, 4(3):267-276, 1968.

4. Raju, I. S. and Newman, J. C., Jr. "Improved Stress Intensity Factors for Semi-Elliptical Surface Cracks in Finite-Thickness Plates", NASA TMX-72825, Fourth International Conference on Structural Mechanics in Reactor Technology, San Francisco, August, 1977.

5. Standard Test Method for Plane-Strain Fracture Toughness of Metallic Materials, ANSI/ASTM, E399-78.

6. Private Communication from J. G. Kaufman, Chairman of ASTM Committee E-24 on Fracture Testing to W. F. Brown, Jr., NASA Lewis Research Center, Cleveland, Ohio.

7. Private Communication from W. F. Brown, Jr. to J. G. Kaufman.

8. Cruse, T. A. and Meyers, G. "Three Dimensional Fracture Mechanics Analysis", Proc. ASCE. J. Struct. Div., 103(ST2):309-320, Feb., 1977.

9. Tracey, D. M. "Finite Elements for Three Dimensional Elastic Crack Analysis", Nucl. Eng. & Des., 26(2):282-290, 1974.

10. Raju, I. S. and Newman, J. C., Jr. "Three Dimensional Finite Element Analysis of Finite Thickness Fracture Specimens", NASA TN D-8414, 1977.

11. Blackburn, W. S. and Hellen, T. K. "Determination of Stress Intensity Factors for Battelle Benchmark Geometries", International Journal of Fracture, 16(5), 1980.

12. McGowan, J. J. "A Photoelastic Investigation of the Effect

of Crack Front Curvature on the Stress Intensity Factor Distribution in a Single Edge Notch Specimen", BER 242-97, Sept., 1979.

13. Fourney, M. E. "Experimental Determination of the Effect of Crack Front Curvature in an ASTM Compact Tension Specimen", Proceedings of the Brazilian Congress of Mechanical Engineering, pp. 13-26, 1977.

14. Pereira, M. F. S., Head, J. L. and Turner, C. E. "Three-Dimensional Crack Analysis Using Standard Isoparametric Elements", Department of Mechanical Engineering Report, Imperial College, London, 1976.

15. Neale, B. K. "The Influence of Crack Shape on Fracture Toughness Testing", International J. of Fracture, 12(3):499-502, 1976.

16. Neale, B. K. "The Stress Intensity Factors Associated with Curved Crack Fronts in a Compact Tension Specimen", Proc. of the First International Conference on Numerical Methods in Fracture Mechanics, Ed. A. R. Luxmoore and D. R. J. Owen, pp. 218-224, 1978.

17. Liskovets, O. A. "The Method of Lines (Review)", Differential Equations, 1:1308-1323, 1965.

18. Jones, D. J., South, J. C., Jr. and Klunker, E. B. "On the Numerical Solution of Elliptic Partial Differential Equations by the Method of Lines", Journal of Computational Physics, 9:496-527, 1972.

19. Kurtz, L. A., Smith, R. E., Parks, C. L. and Boney, L. R. "A Comparison of the Method of Lines to Finite Difference Technique in Solving Time - Dependent Partial Differential Equations", Computer and Fluids, 6:49-70, 1978.

20. Hopkins. T. R. and Wait, R. "A Comparison of Galerkin, Collocation and the Method of Lines for Partial Differential Equations", International Journal of Numerical Methods in Engineering, 12:1081-1107, 1978.

21. Gyekenyesi, J. P. and Mendelson, A. "Three Dimensional Elastic Stress Analysis of Finite Geometry Solids Containing Cracks", Int. J. of Fracture, 11(3), 1975.

22. Gyekenyesi, J. P. and Mendelson, A. "Stress Analysis and Stress Intensity Factors for Finite Geometry Solids Containing Rectangular Surface Crack", Trans. of the ASME, J. Appl.

Mech., pp. 442-448, 1977.

23. Fu, L. S. and Malik, S. N. "The Method of Lines Applied to Crack Problems Including Plasticity Effects", Computer and Structures, 10:447-456, 1979.

24. Malik, S. N. and Fu, L. S. "Elasto-Plastic Analysis for a Finite Thickness Rectangular Plate Containing a Through-Thickness Central Crack", Int. J. of Fracture, 18:45-63, 1982.

25. Gross, B. "Some Plane Problem Elastostatic Solutions for Plates Having a V-Notch", Ph.D. Thesis, Case Western Reserve University, Cleveland, Ohio 44106.

26. Mendelson, A. Plasticity: Theory and Application, The MacMillan Co., New York, 1968.

27. Rice, J. R. "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks", Trans. of the ASME, J. Appl. Mech., 34:379-386, 1968.

28. Parks, D. M. "A Stiffness Derivative Finite Element Technique for Determination of Crack Tip Stress Intensity Factors", Int. J. of Fracture, 10(4), 1974.

29. Hellen, T. K. "On the Method of Virtual Crack Extension", International Journal for Numerical Methods in Engineering, 9:187-207, 1975.

30. Blackburn, W. S. and Hellen, T. K. "Calculation of Stress Intensity Factors in Three Dimensions by Finite Element Methods", International Journal for Numerical Methods in Engineering, 11:211-229, 1977.

31. Private Communication from J. P. Gyekenyesi, NASA Lewis Research Center, Cleveland, Ohio.

32. Private Communication from I. S. Raju, JIAFS, George Washington University and NASA Langley Research Center, Hampton, VA.

33. Hulbert, L. E., Ed., Three Dimensional Fracture Analysis, Battelle, Columbus, Ohio, 1976. Also, Int. J. of Fracture, 13:RCR87-91. 1977.

34. Private Communication from W. F. Brown, Jr., NASA Lewis Research Center, Cleveland, Ohio.

35. Paris, P. C., Tada, H., Zahoor, A. and Ernst, H. "The Theroy of Instability of the Tearing Mode of Elastic-Plastic Crack

Growth", Elastic-Plastic Fracture, ASTM STP668, J. D. Landes, J. A. Begley, and G. A. Clarke, Eds., American Society for Testing and Materials, pp. 5-36, 1979.

36. Shiratori, M. and Miyoshi, T. "A Comparison of Finite Element Solutions for J-Integral Analysis of Compact Specimen (A Round Robin Test in Japan)", Numerical Methods in Fracture Mechanics, Proc. of the Second International Conference, University College, Swansea, pp. 417-431, Eds., D. R. J. Owen and A. R. Luxmoore, 1980.

37. Miyamoto, H. and Kikuchi, M. "Evaluation of Three Dimensional J-Integral of the CT Specimen in Elasto-Plastic States", Trans. of the 6th International Conference on Structural Mechanics in Reactor Technology, Paris, L6/4, 1981.

38. Parks, D. M. "Some Problems in Elastic-Plastic Finite Element Analysis of Cracks", Ph.D. Dissertation, Brown University, Providence, R. I., 1975.

39. McMeeking, R. M. "Finite Deformation Analysis of Crack Tip Opening in Elastic-Plastic Materials and Implications for Fracture Initiations", Technical Report E(11-1), 3084/44, Division of Engineering, Brown University, Providence, R. I., 1976.

40. Standard Test Method for Plane-Strain Fracture Toughness of Metallic Materials, ASTM, E399-81.

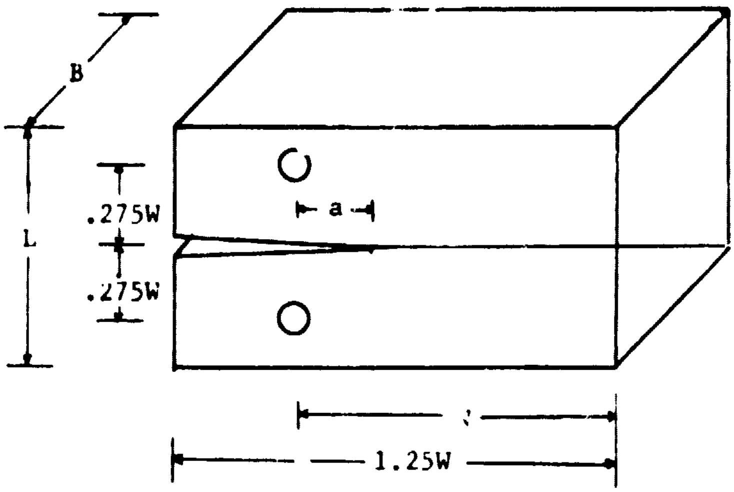

Figure 1(a). Standard Compact Tension Specimen.

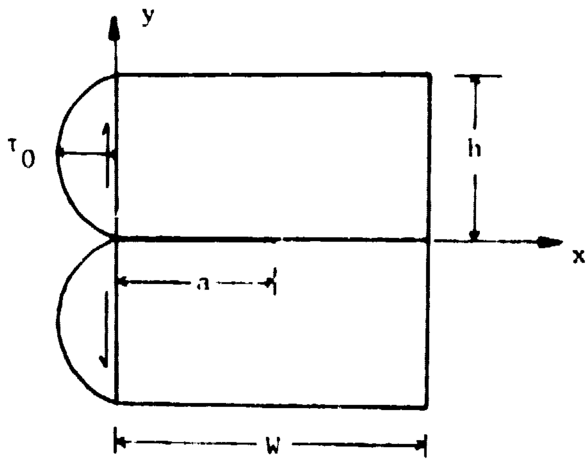

Figure 1(b). Idealized model of Standard Compact Tension Specimen.

Figure 1. Standard Compact Tension Specimen and its idealized model.

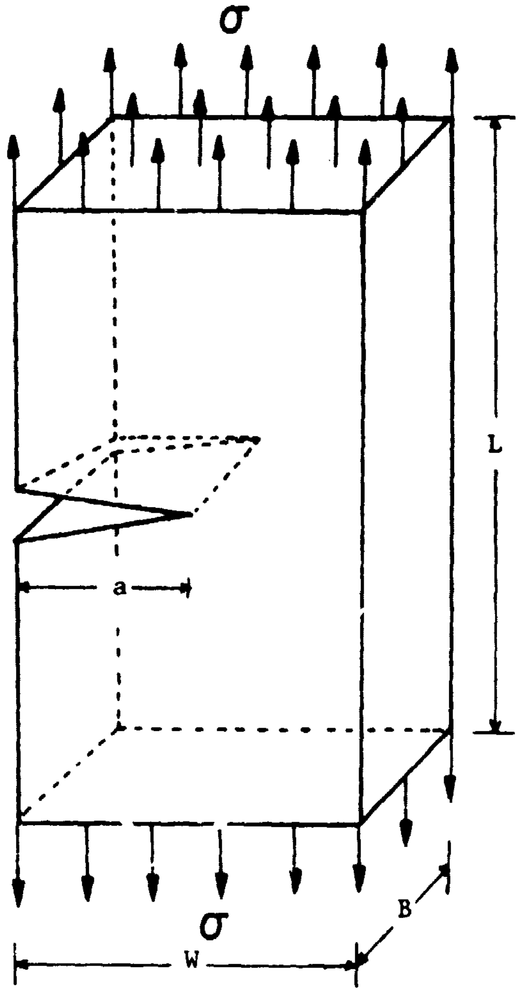

Figure 2. Single Edge-Notched specimen with tensile loading.

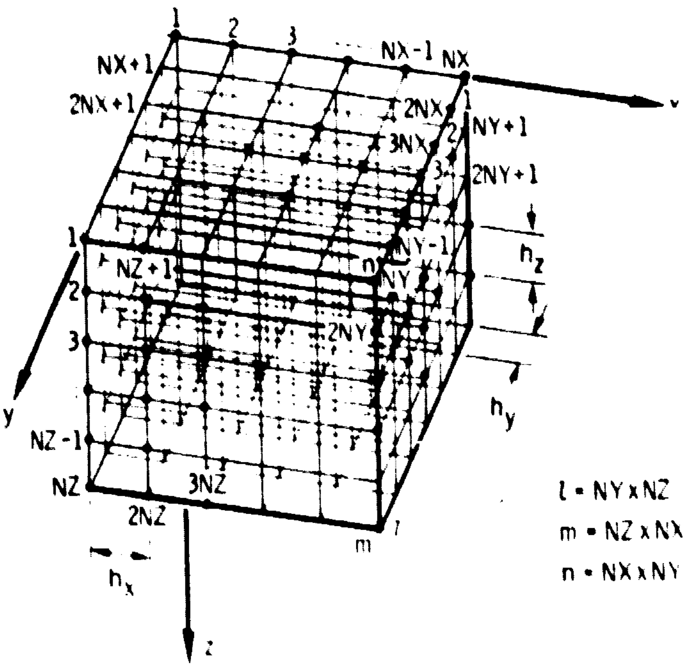

Figure 3. Sets of lines parallel to Cartesian coordinates.

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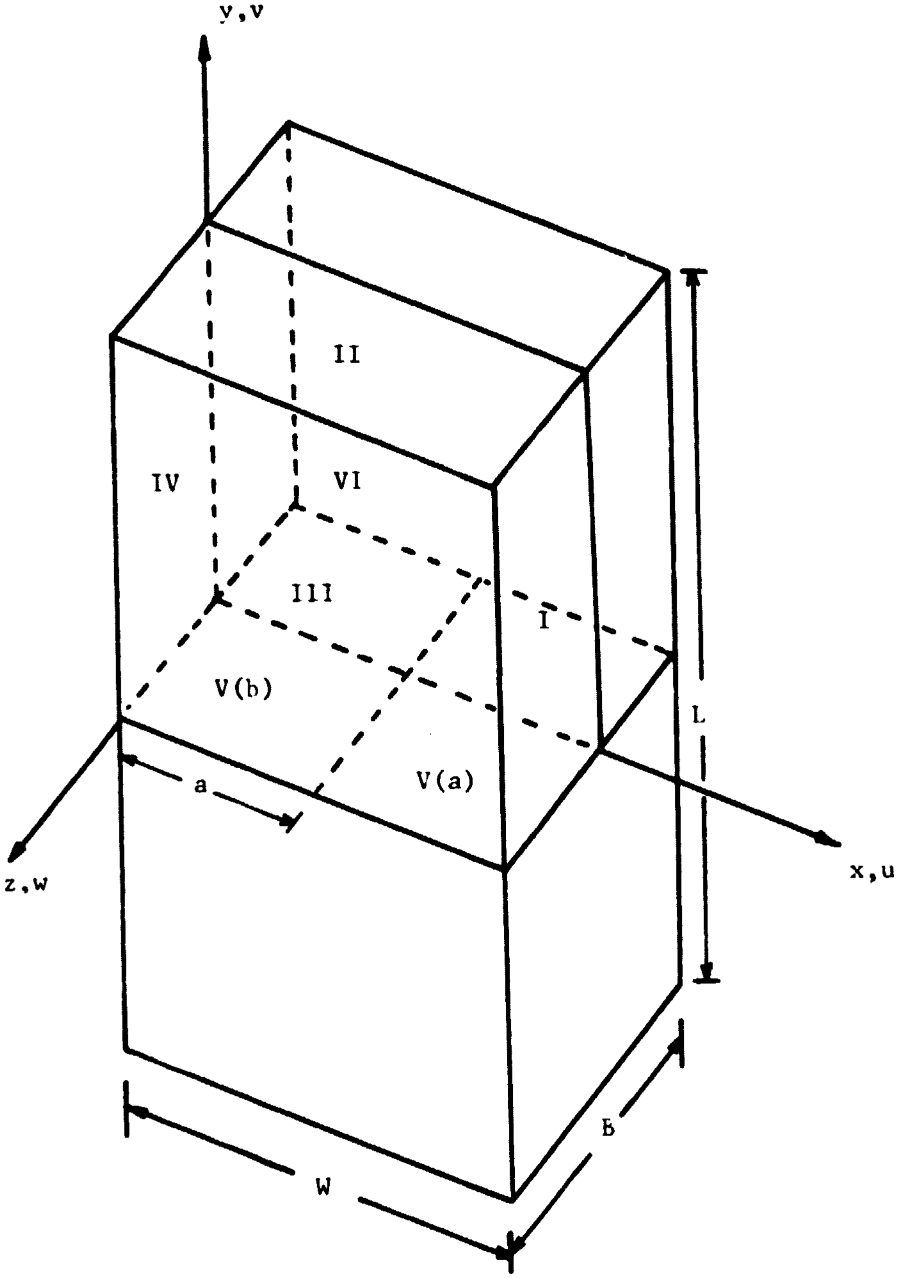

Figure 4. Six Faces of one quarter part of the complete specimen.

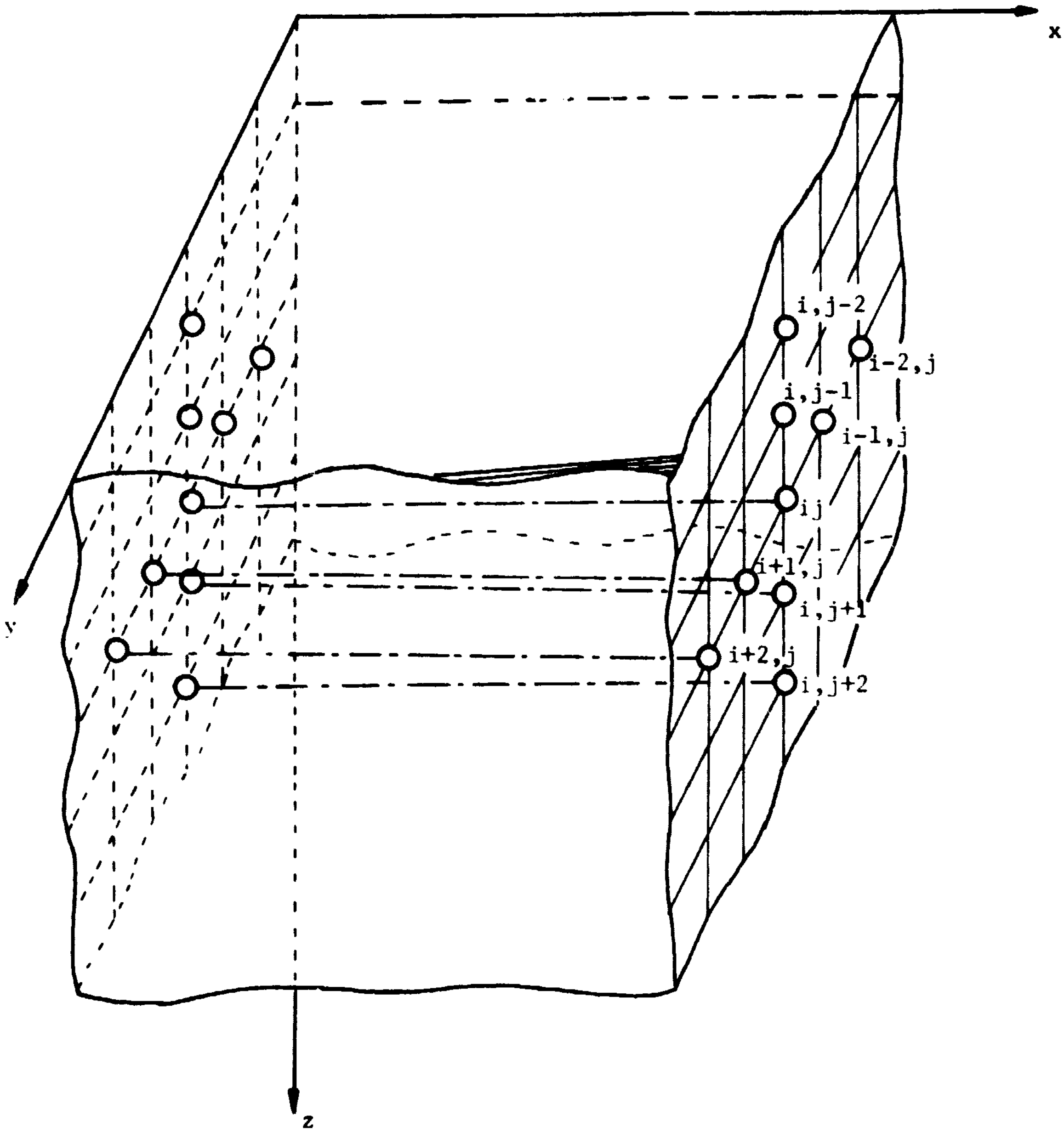

Figure 5. Set of interior lines parallel to x-coordinate.

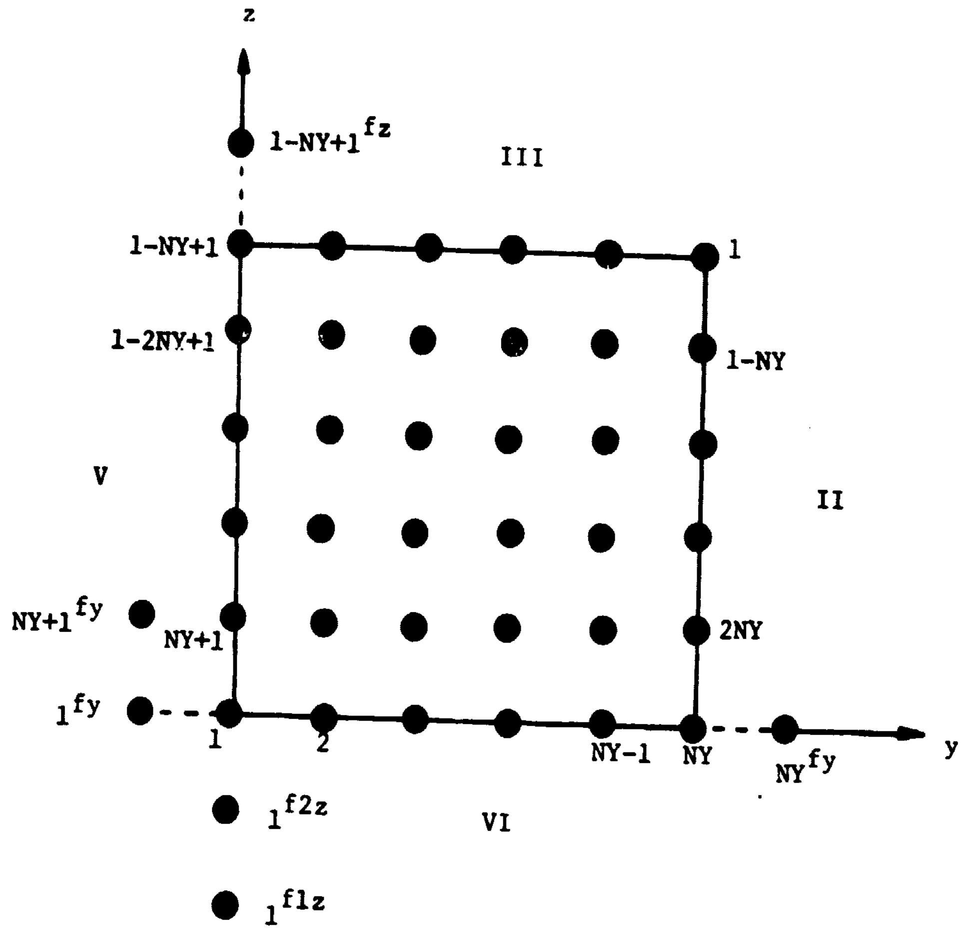

Figure 6. Set of lines parallel to x-coordinate and perpendicular to coordinate plane y-z, l=NYxNZ

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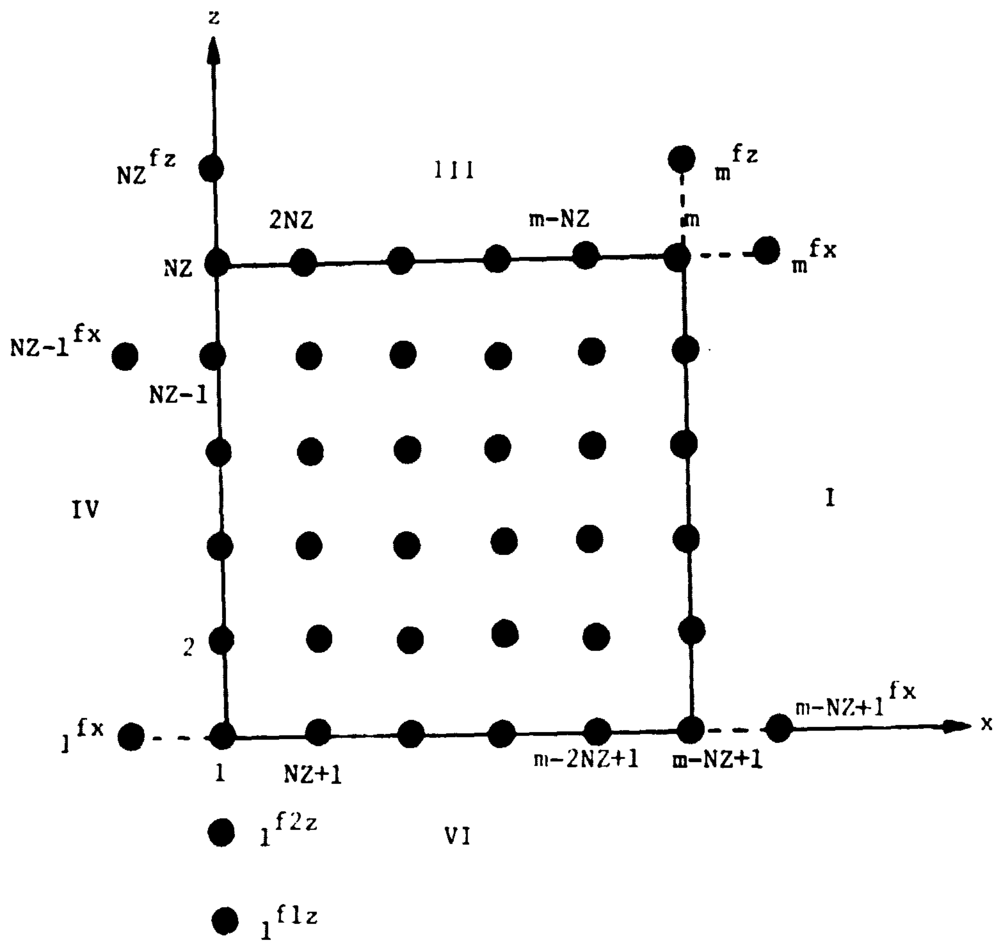


Figure 7. Set of lines parallel to y-coordinate and perpendicular to coordinate plane x-z, m=NZxNX

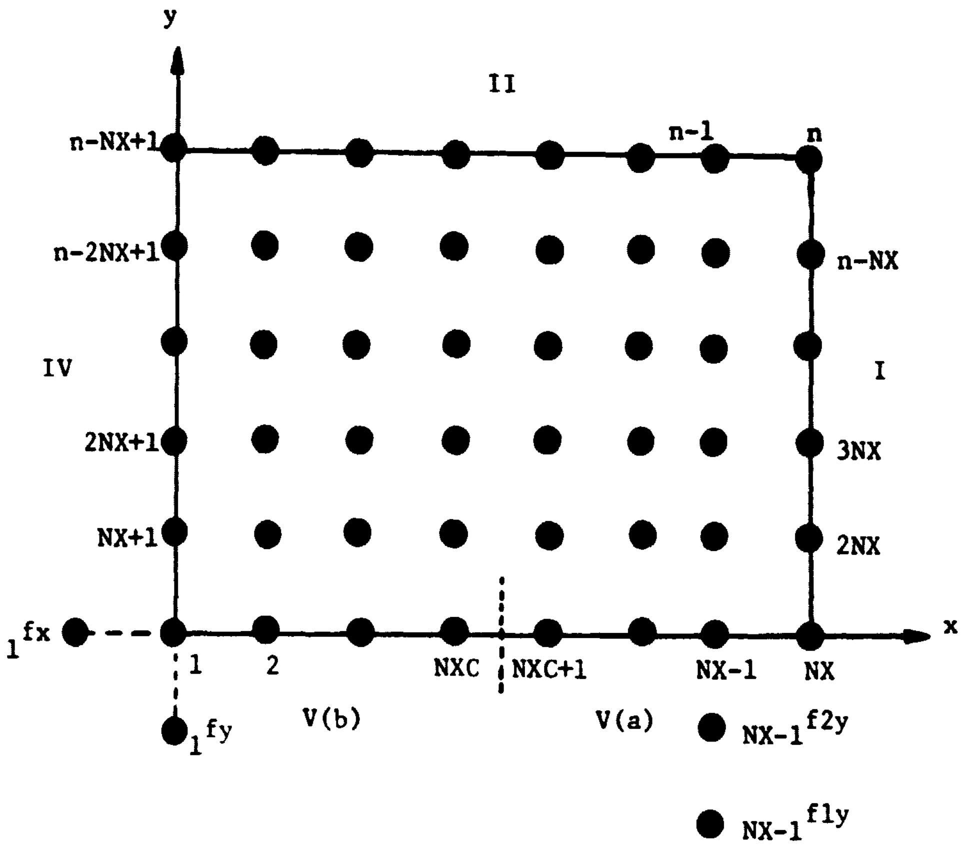
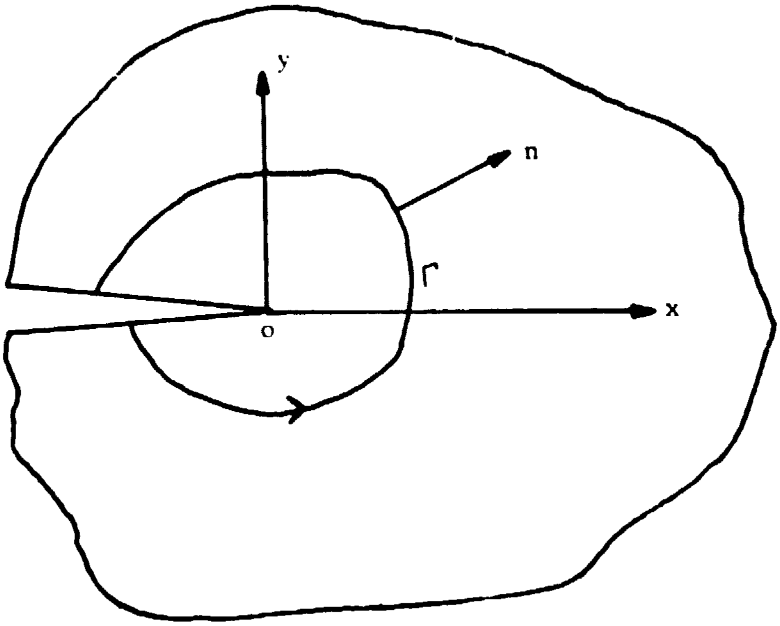
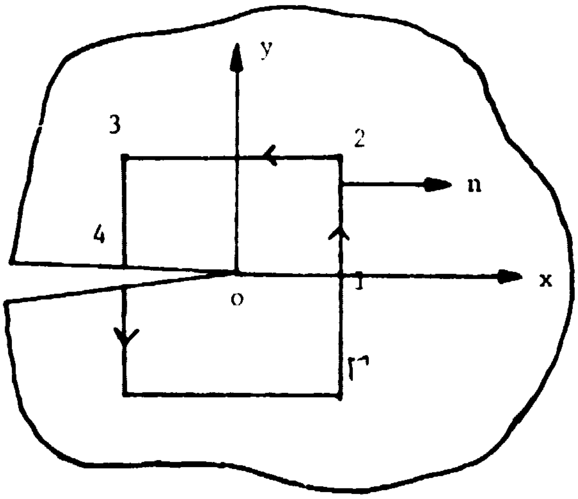


Figure 8. Set of lines parallel to z-coordinate and perpendicular to coordinate plane x-y, n=NYxNX.



a) An arbitrary contour in an non-linear elastic body containing a crack.



b) A rectangular path with coordinate numbering.

Figure 9. Path for the determination of J-integral.

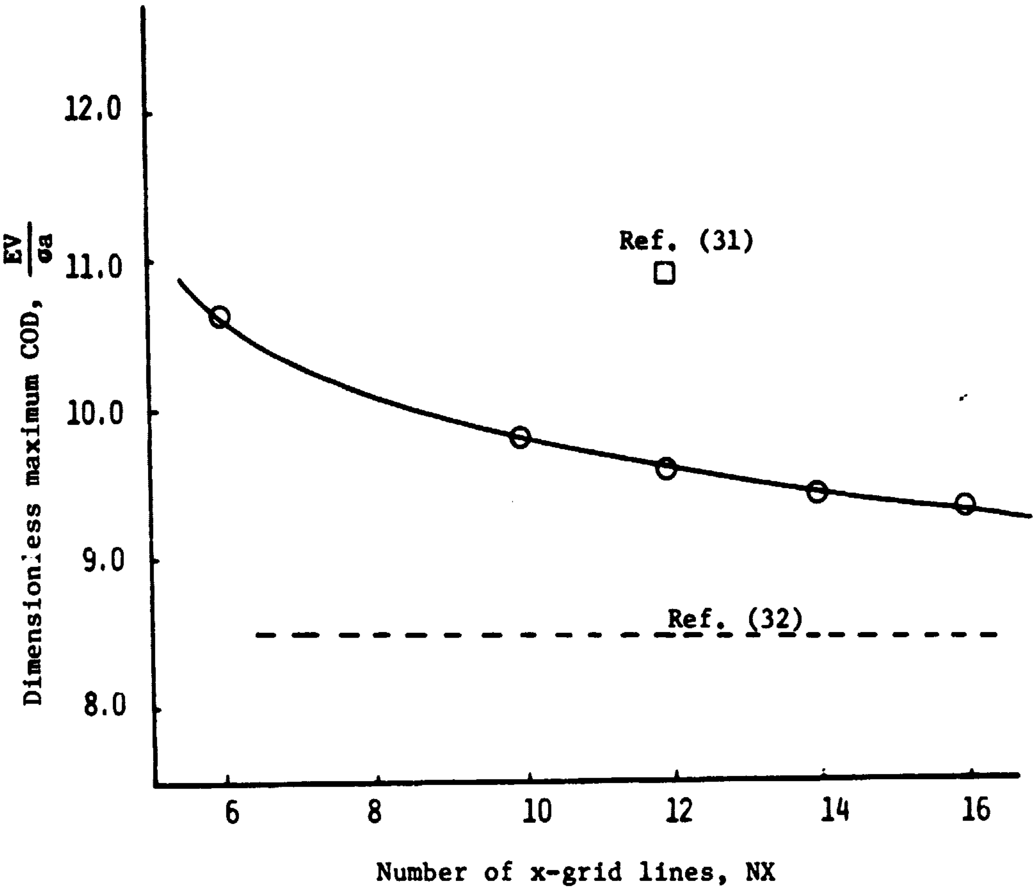


Figure 10. Variation of maximum crack opening displacement with number of x-grid lines for single edge-notched tensile specimen, NY=NZ=8.

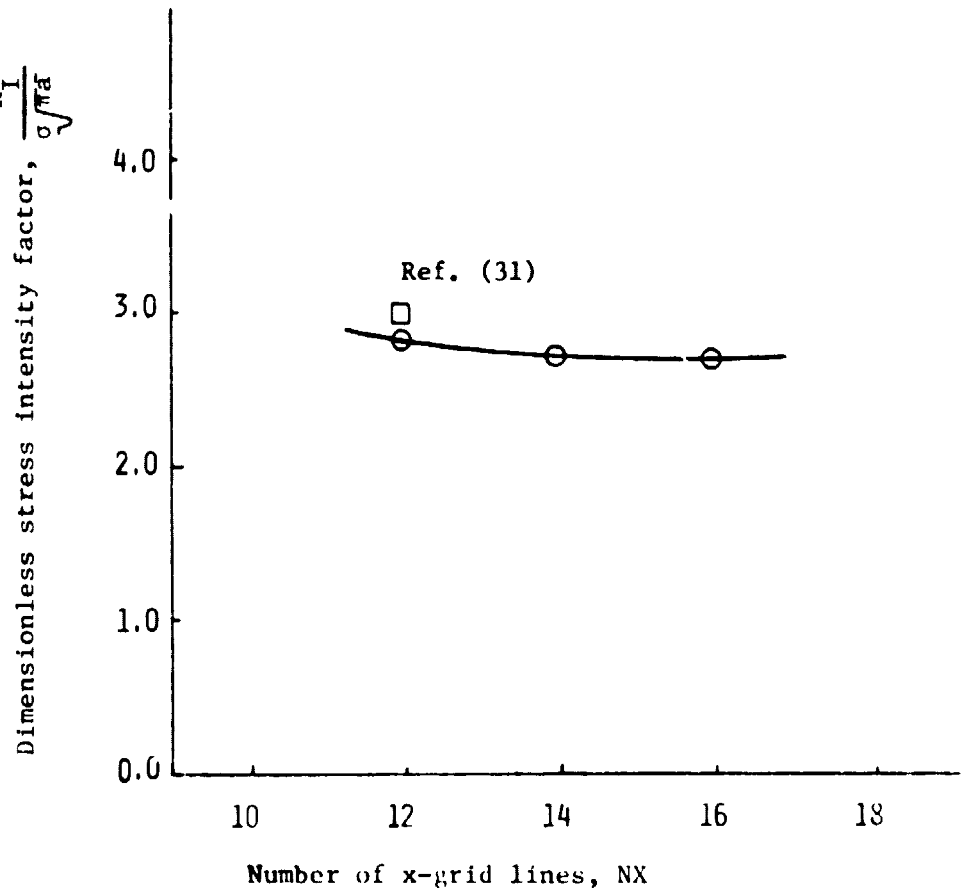


Figure 11. Dimensionless SIF at the center of single edge-notched specimen versus number of x-grid lines, NY=NZ=8.

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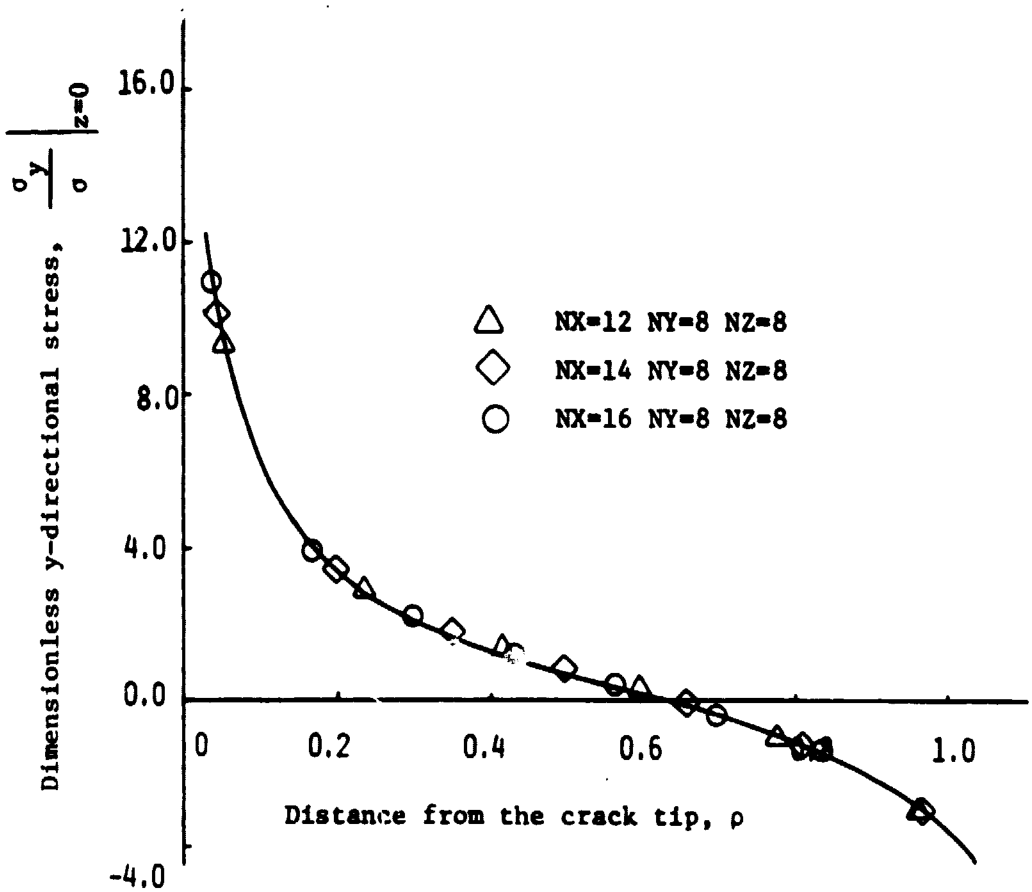

Figure 12. Dimensionless stress versus dimensionless distance from crack tip for various grids, for single edge-notched specimen.

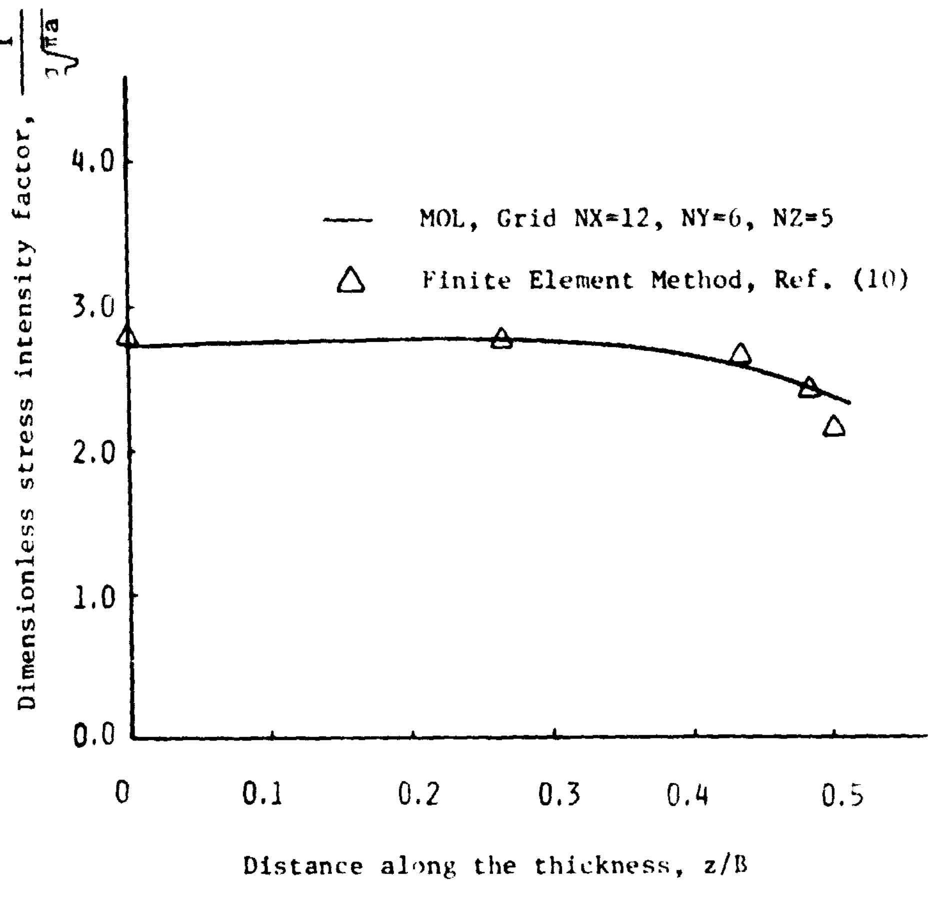

Figure 13. Variation of nondimensional stress intensity factor through the thickness for single edge-notched tensile specimen.

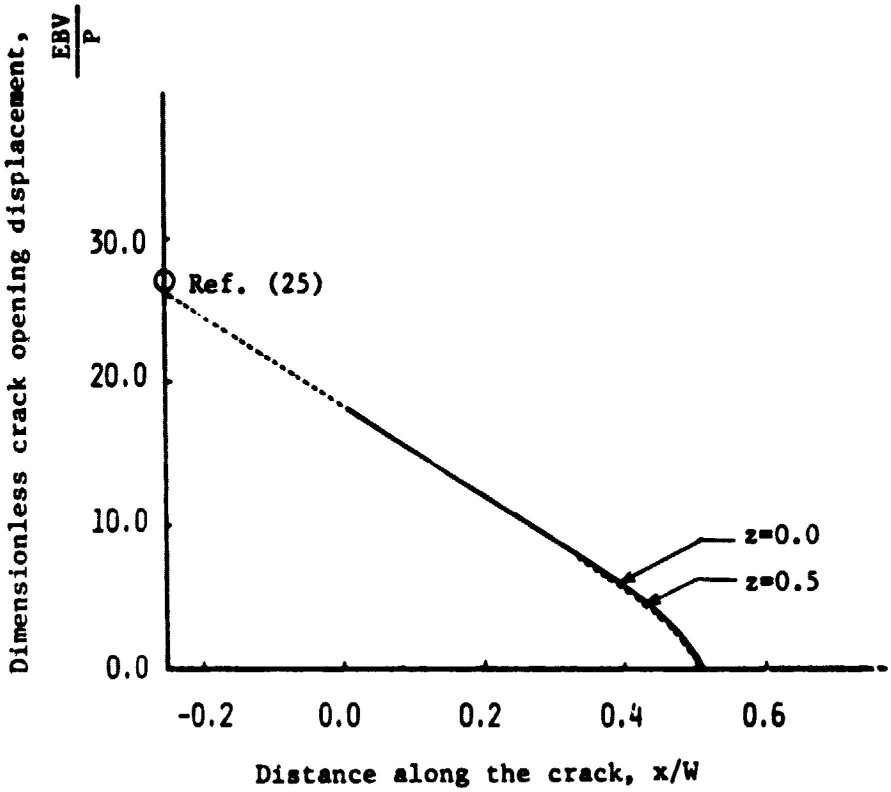

Figure 14. Crack opening displacement along the crack at the middle and the surface of the compact tension specimen.

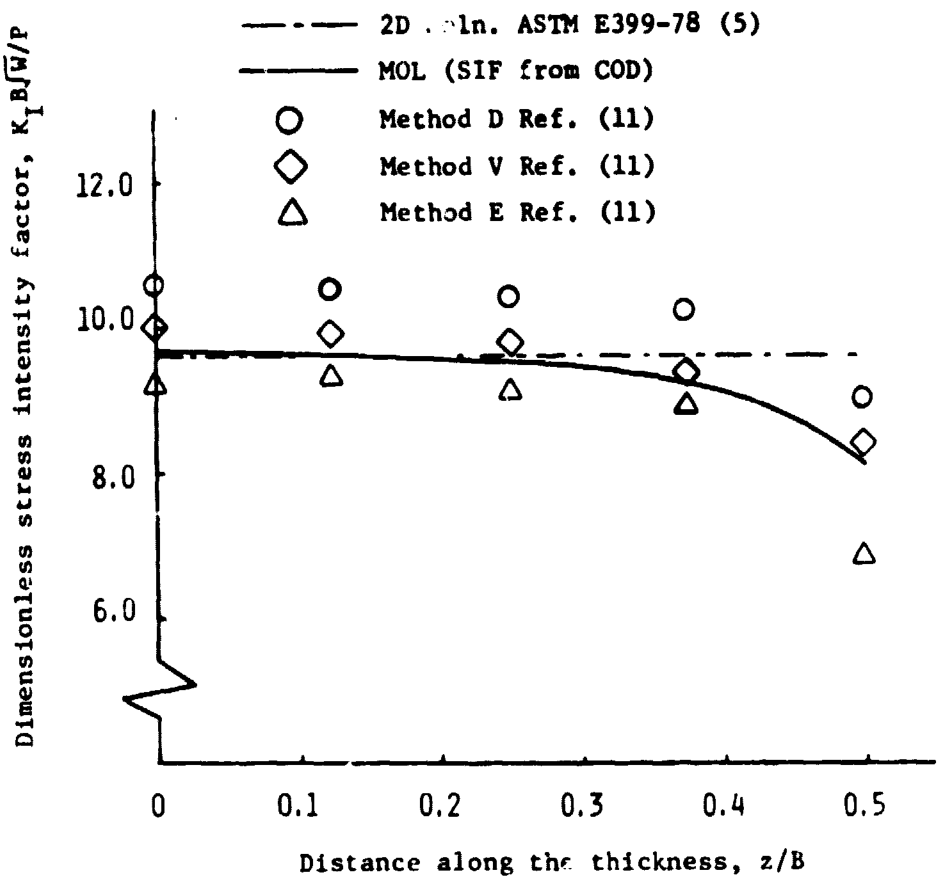

Figure 15. Variation of dimensionless SIF through the thickness of the compact tension specimen.

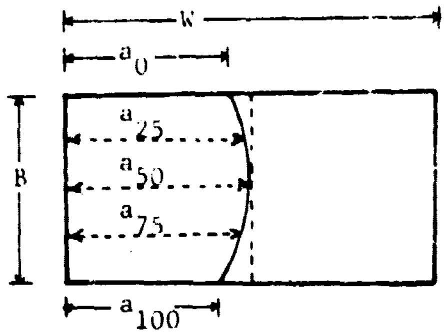

Figure 16(a). Different crack length measurements.

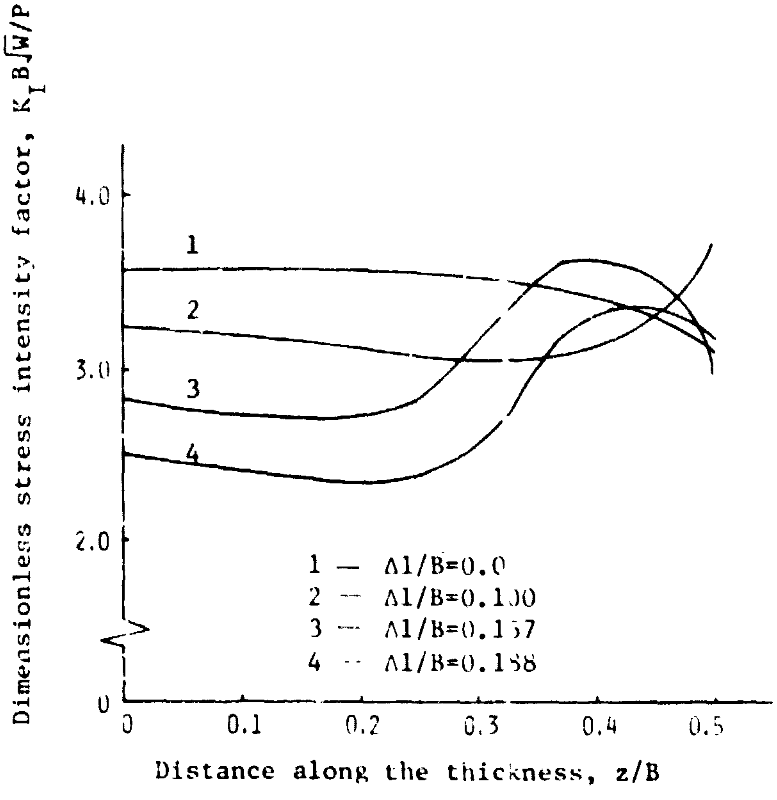

Figure 16. Variation of dimensionless SIF with thickness for different crack tunnel depths for compact tension specimen under tensile loading.

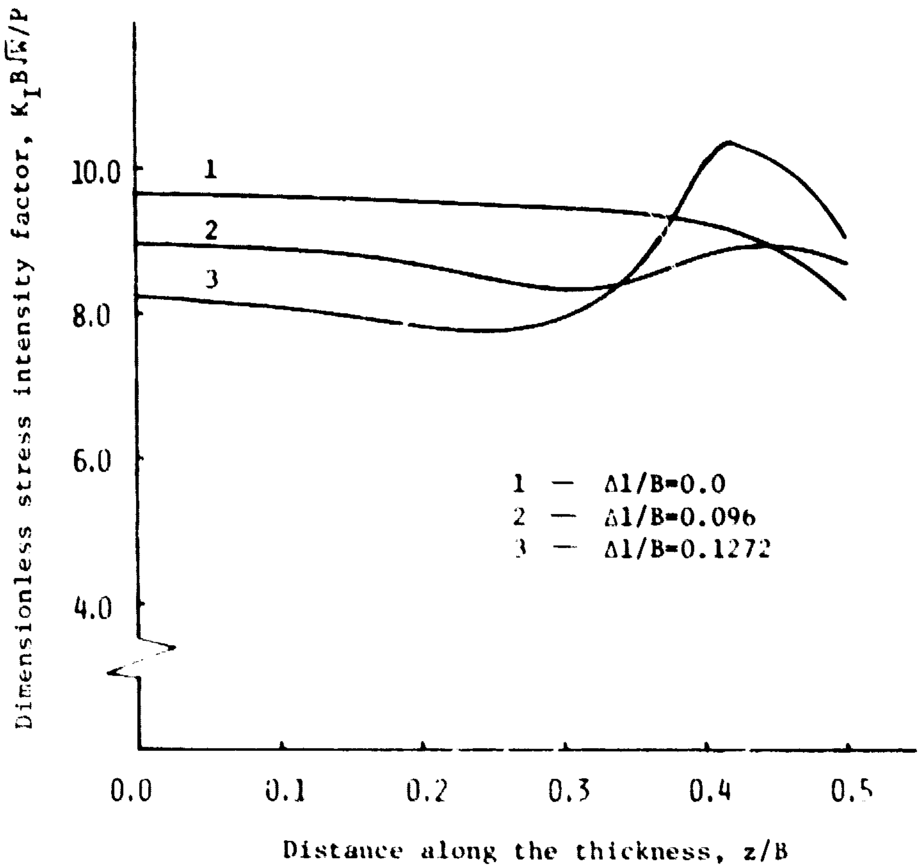

Figure 17. Variation of dimensionless SIF through the thickness for different crack tunnel depths, for compact tension specimen under parabolically applied shear load.

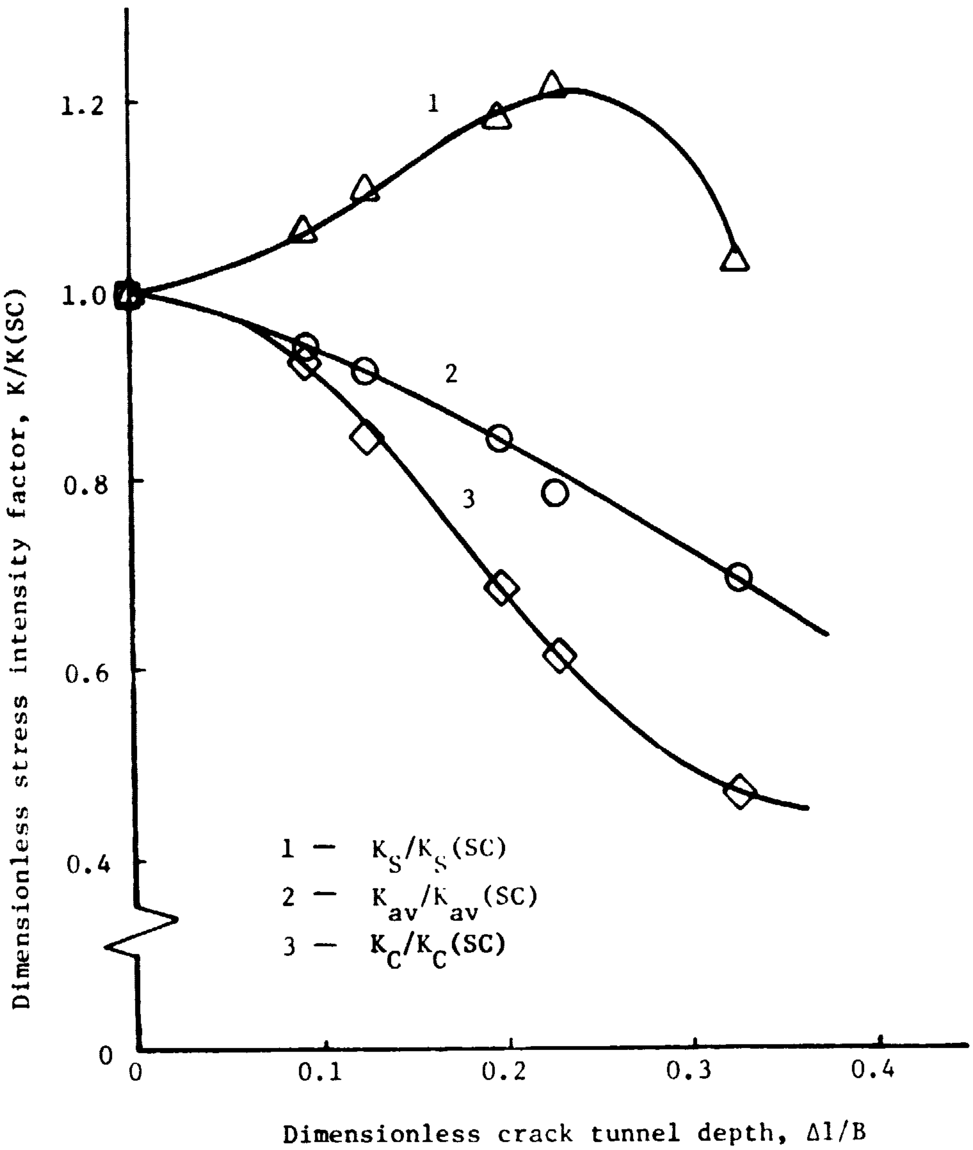

Figure 18. Variation of center, surface and thickness average stress intensity factor with increasing crack tunnel depth.

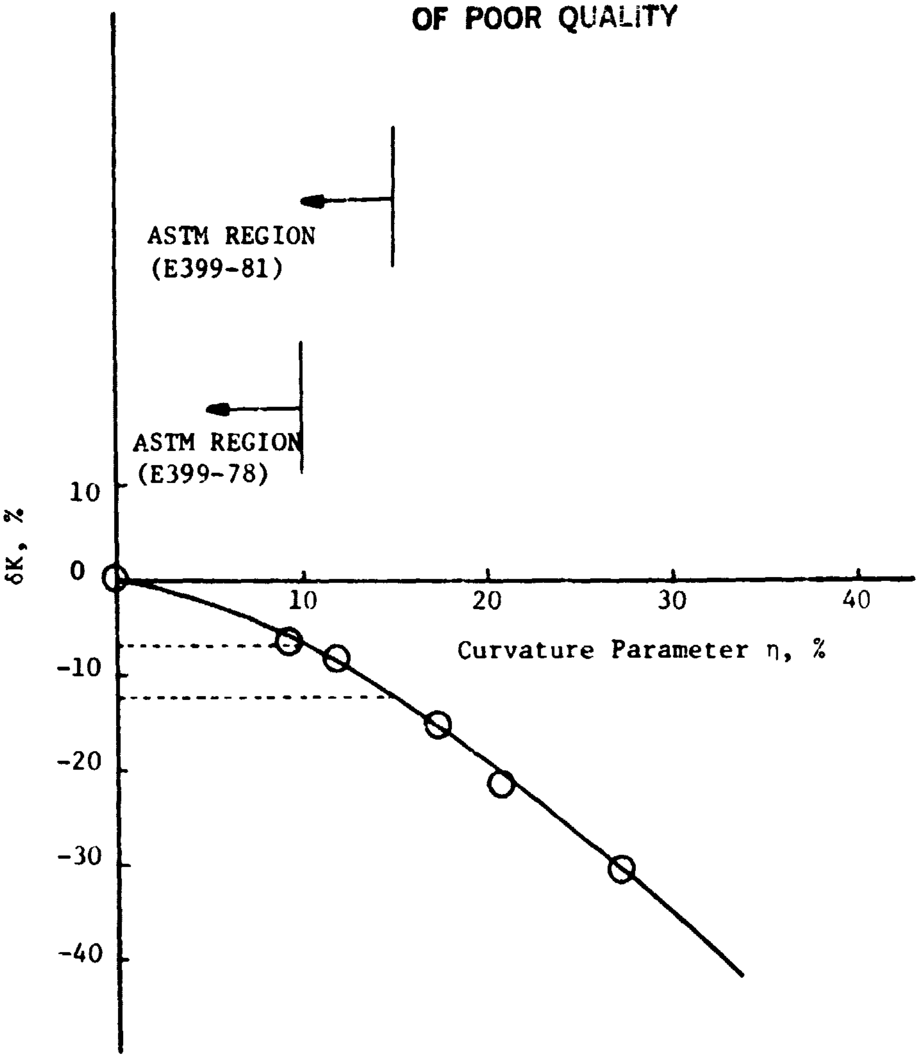

Figure 19. Percent variation in average stress intensity factors between straight crack and curved crack fronts.

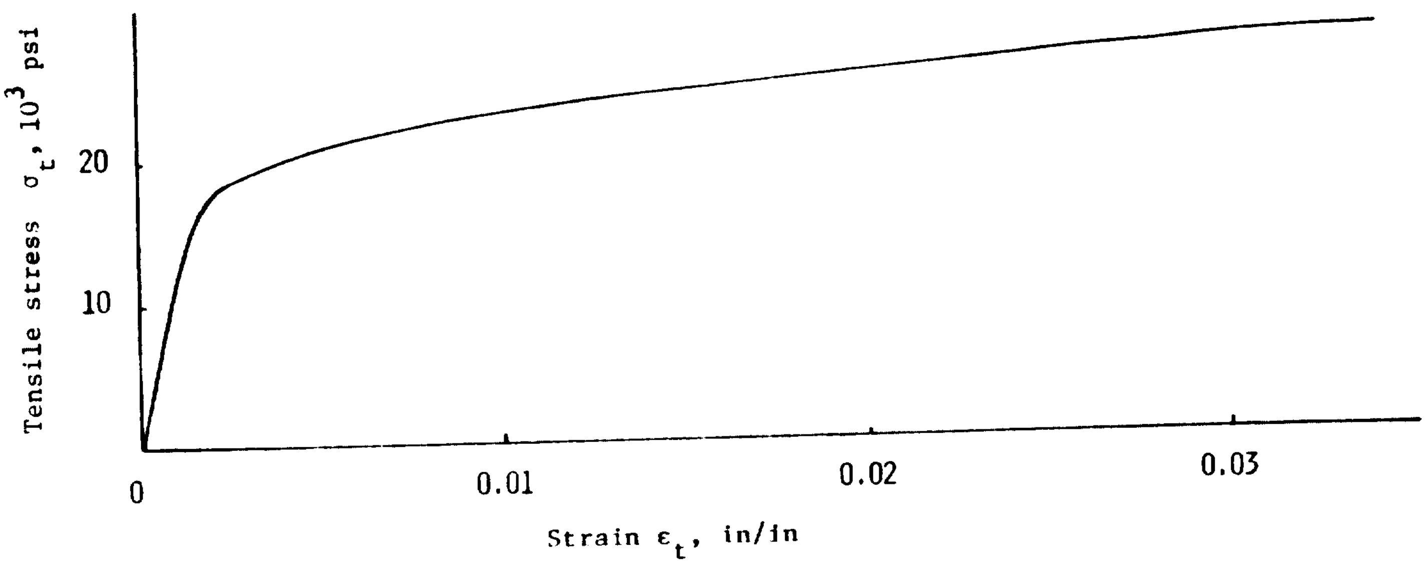

Figure 20. Stress-Strain curve for the Al-5083 material used in experiment.
$E=10.4\times10^6$ lb./in^2, $\nu=0.33$

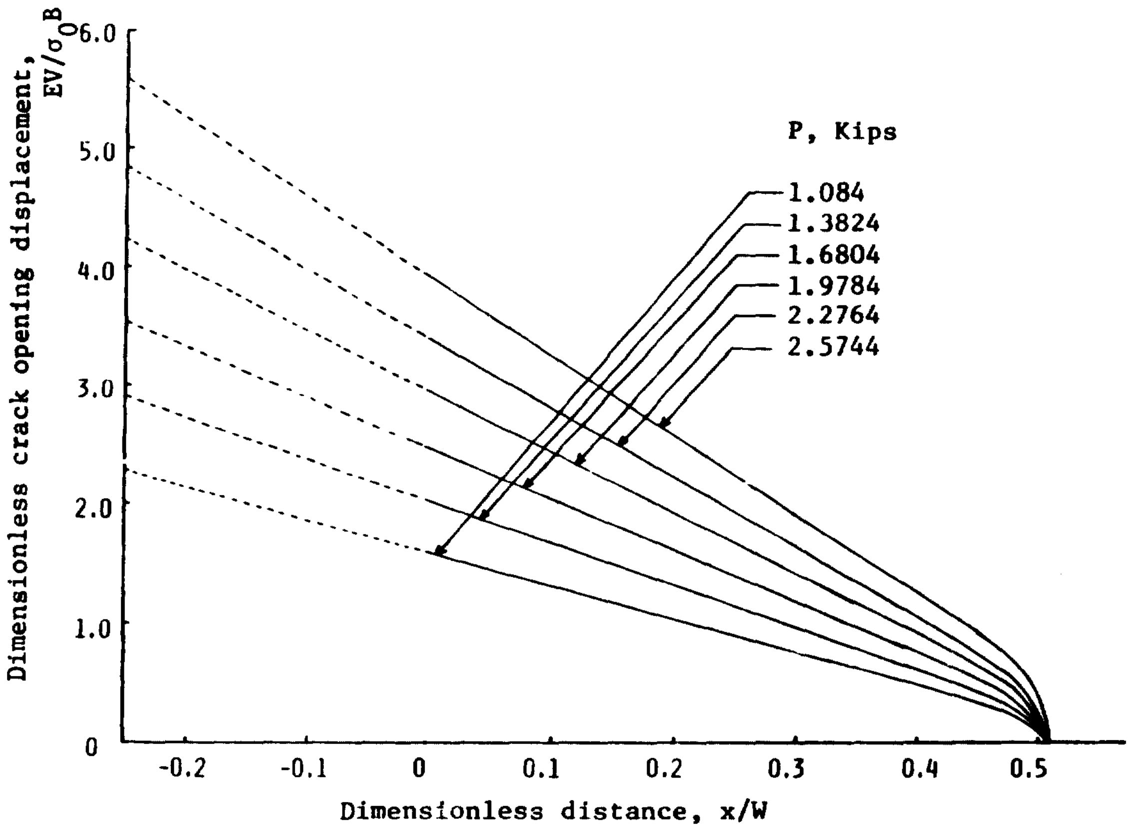

Figure 21. Variation of non-dimensional crack opening displacement along the width of the compact tension specimen.

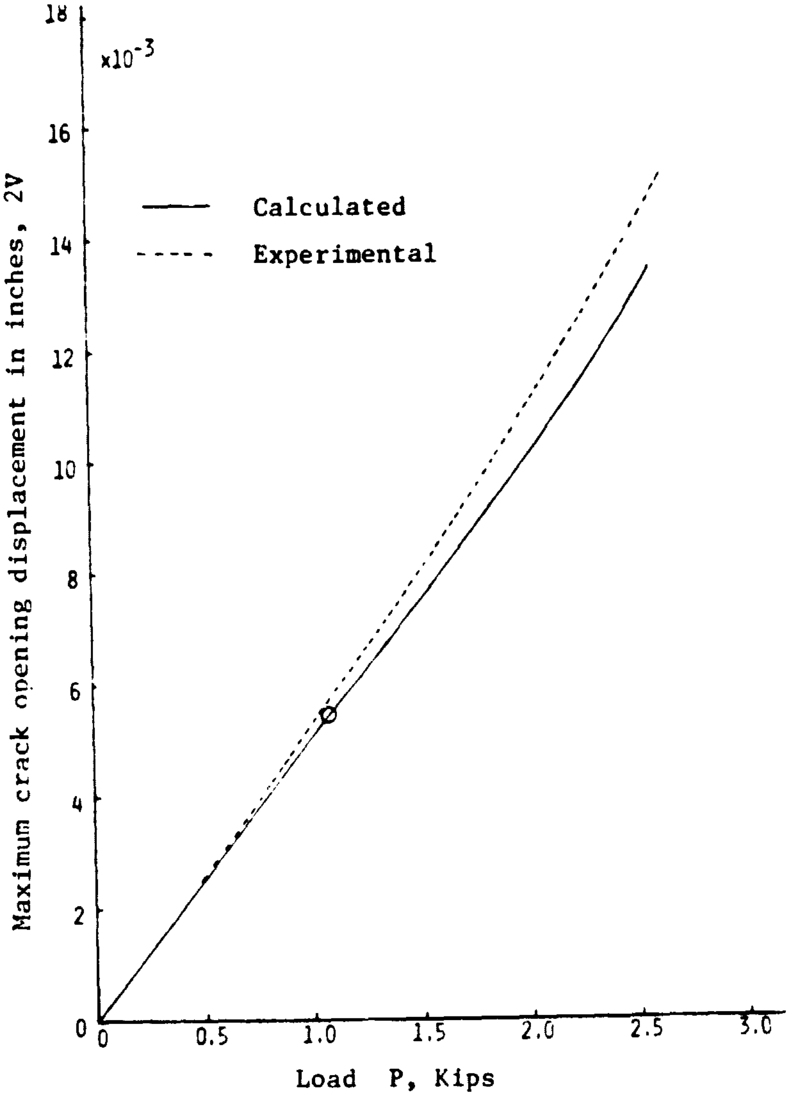

Figure 22. Load versus maximum crack opening displacement for the compact tension specimen.

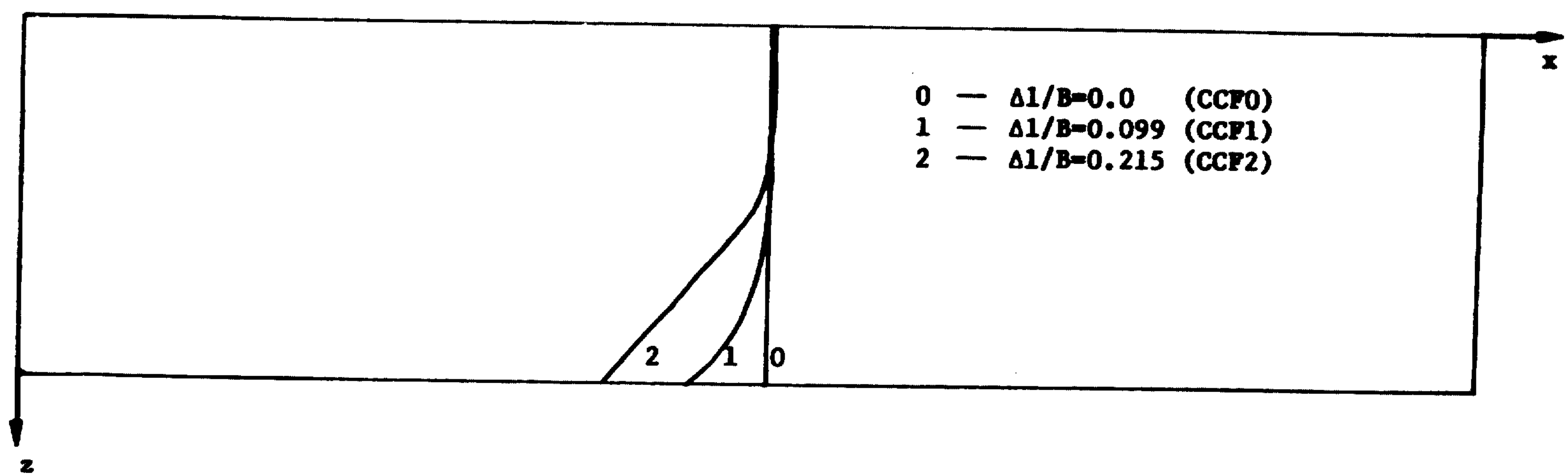

Figure 23. Crack front shapes used in elasto-plastic analysis. Half of specimen shown.

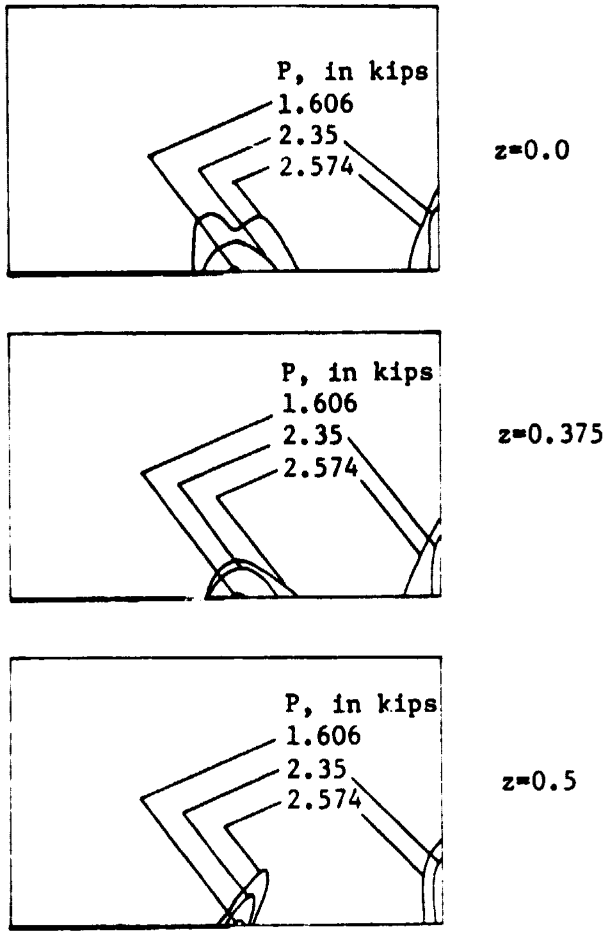

Figure 24. Growth of plastic zone with increasing load for straight crack front at three locations of the CT specimen.

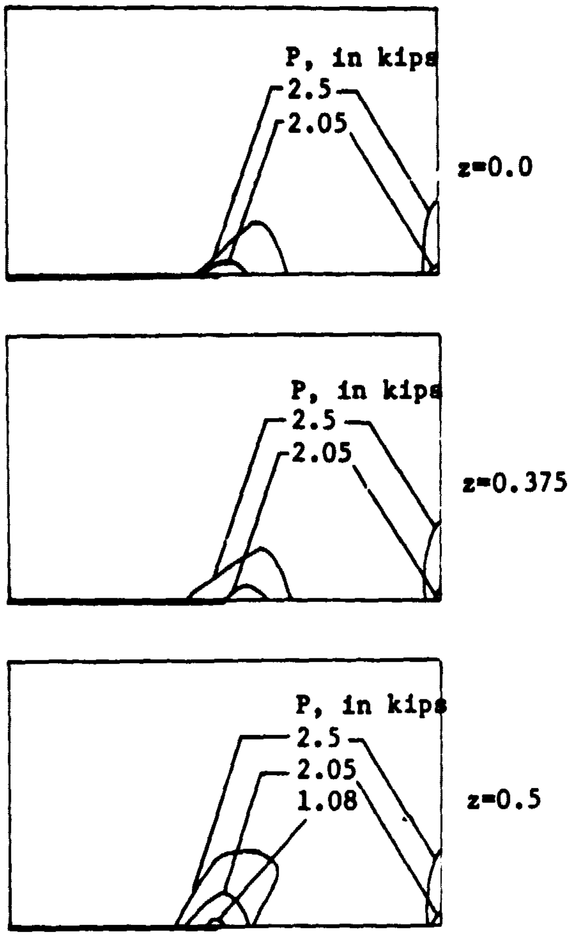

Figure 25. Growth of plastic zone with increasing load for curved crack front 1 at three locations of the CT specimen.

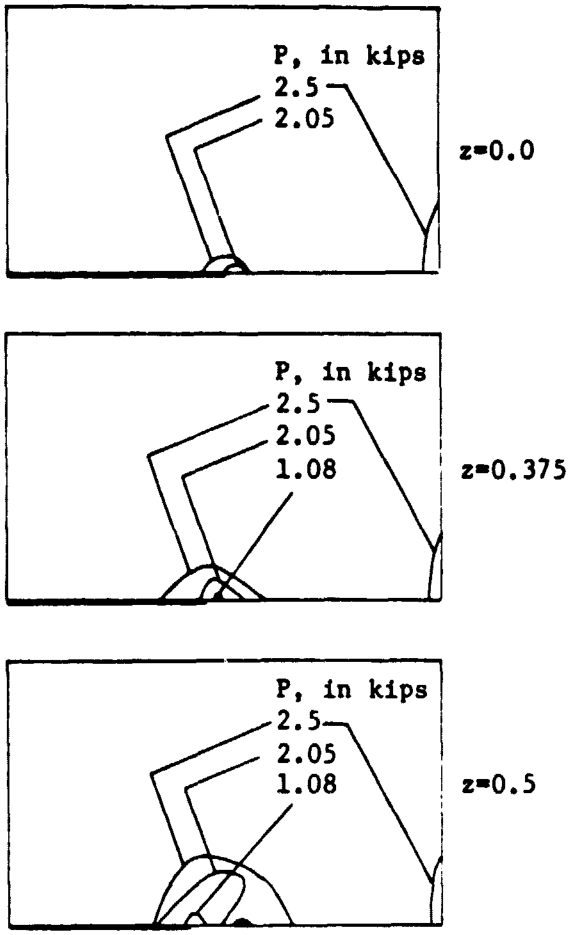

Figure 26. Growth of plastic zone with increasing load for curved crack front 2 at three locations of the CT specimen.

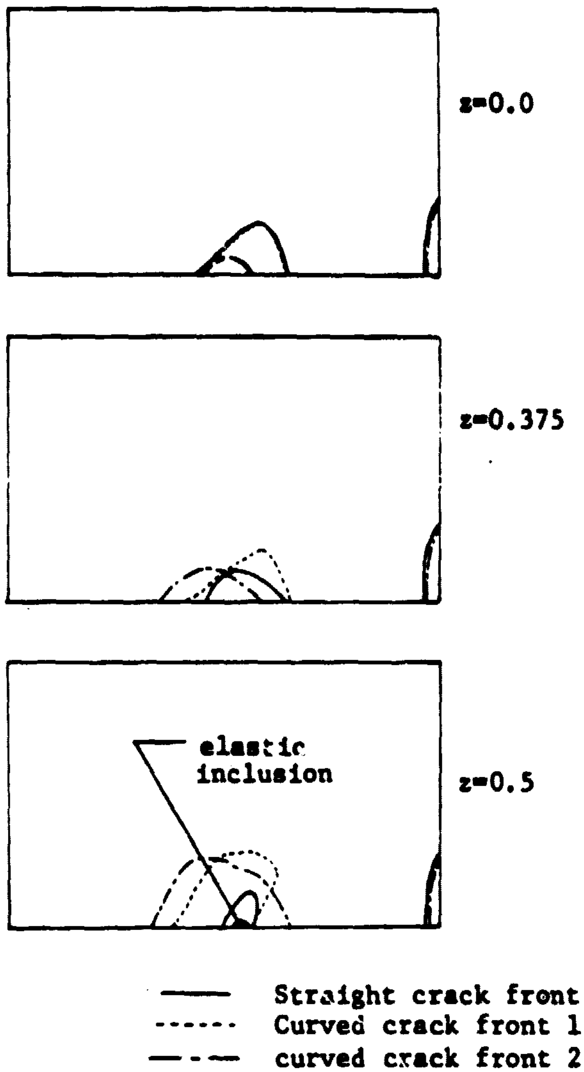

Figure 27. Plastic zone enclaves for different curved crack fronts at three locations of the CT specimen (P=2.5 Kips).

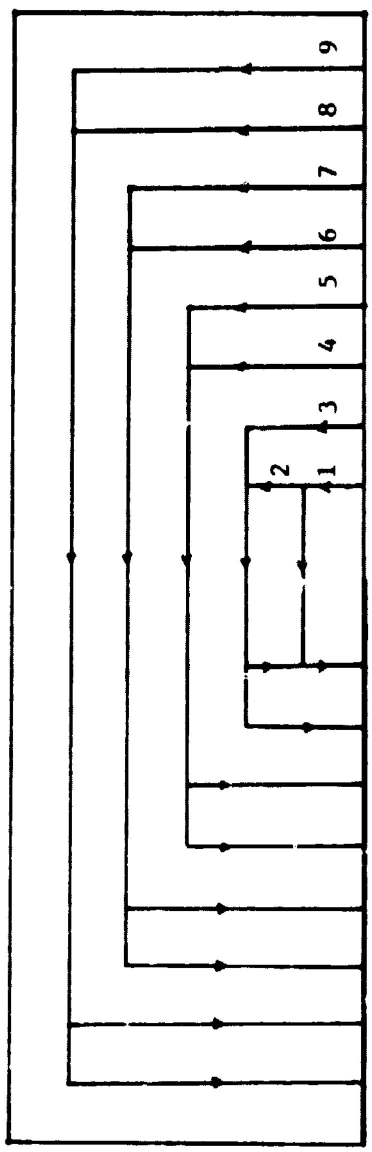

Figure 28. Different paths used for evaluating J-integral for straight crack front.

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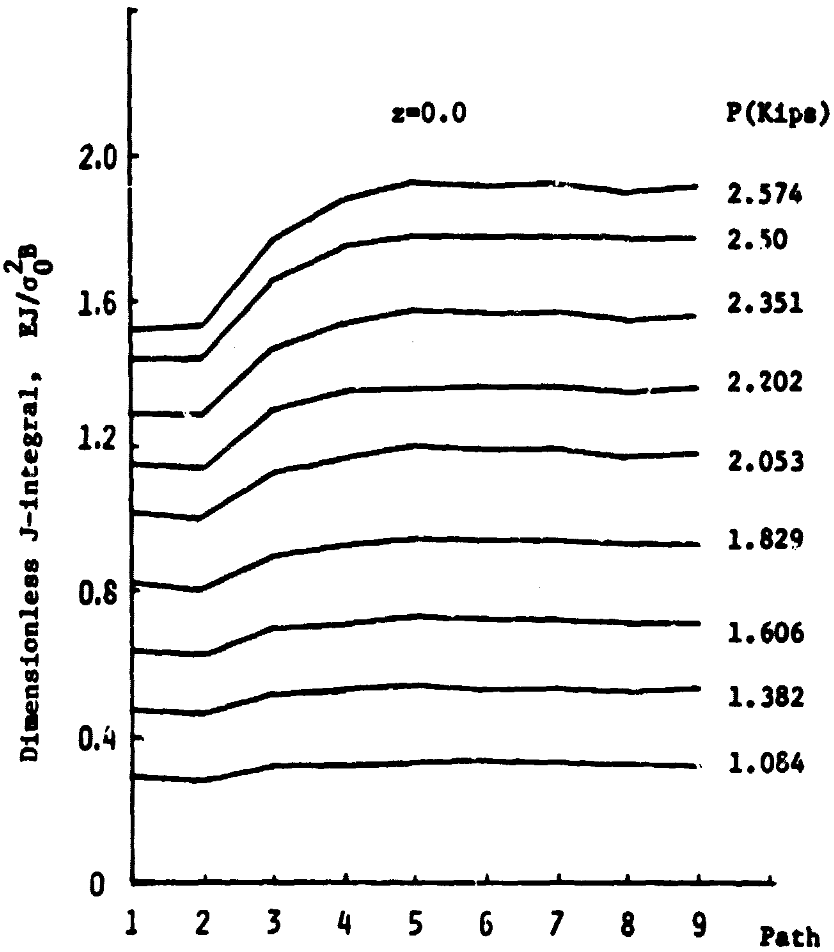

Figure 29. Non-dimensiona' J-integral values for different paths at the center of the CT specimen with a straight crack front for increasing load.

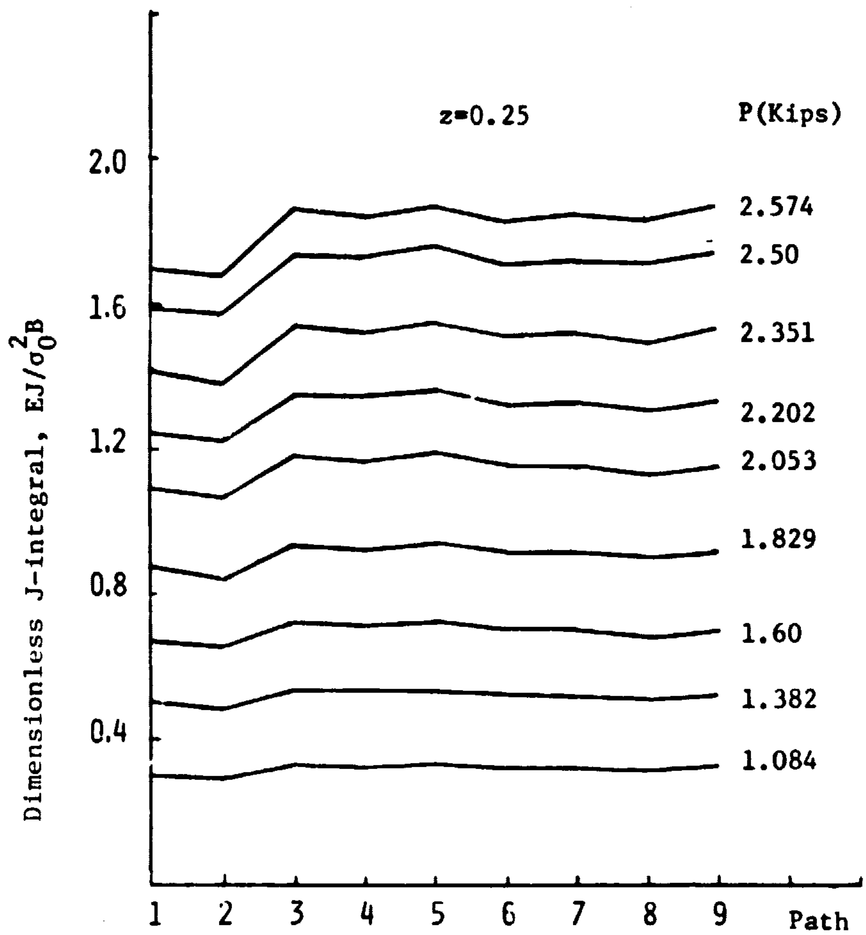

Figure 30. Non-dimensional J-integral values for different paths at midway between center and surface of the CT specimen with a straight crack front for increasing load.

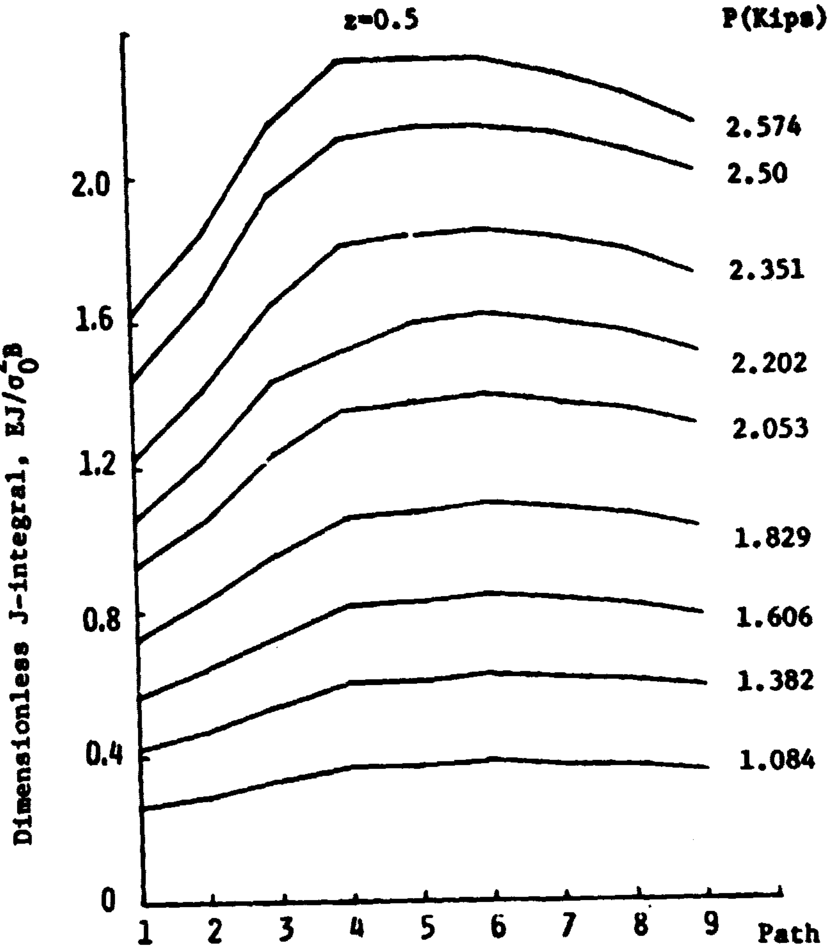

Figure 31. Non-dimensional J-integral values for different paths at the surface of the CT specimen with straight crack front for increasing load.

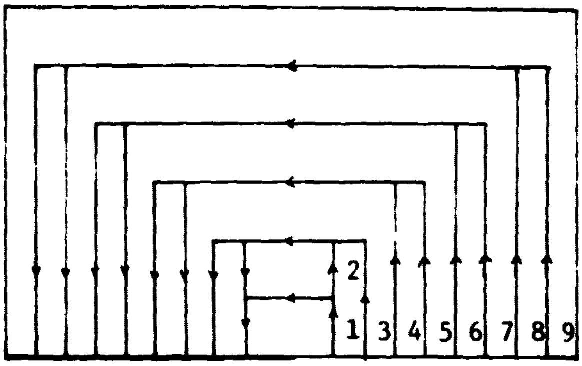

Figure 32(a). Different paths for the plane located at z=0.0 through 0.375

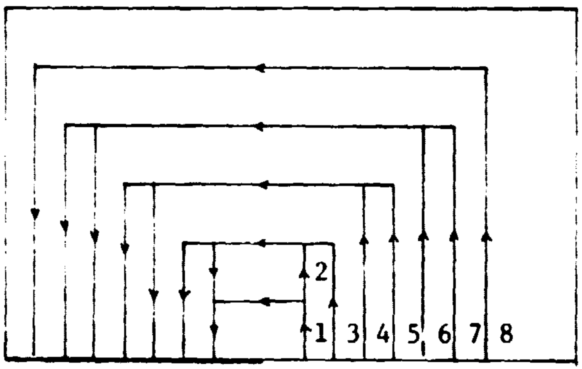

Figure 32(b). Different paths for the plane located at z=0.5

Figure 32. Different paths used for evaluating J-integral for curved crack front 1

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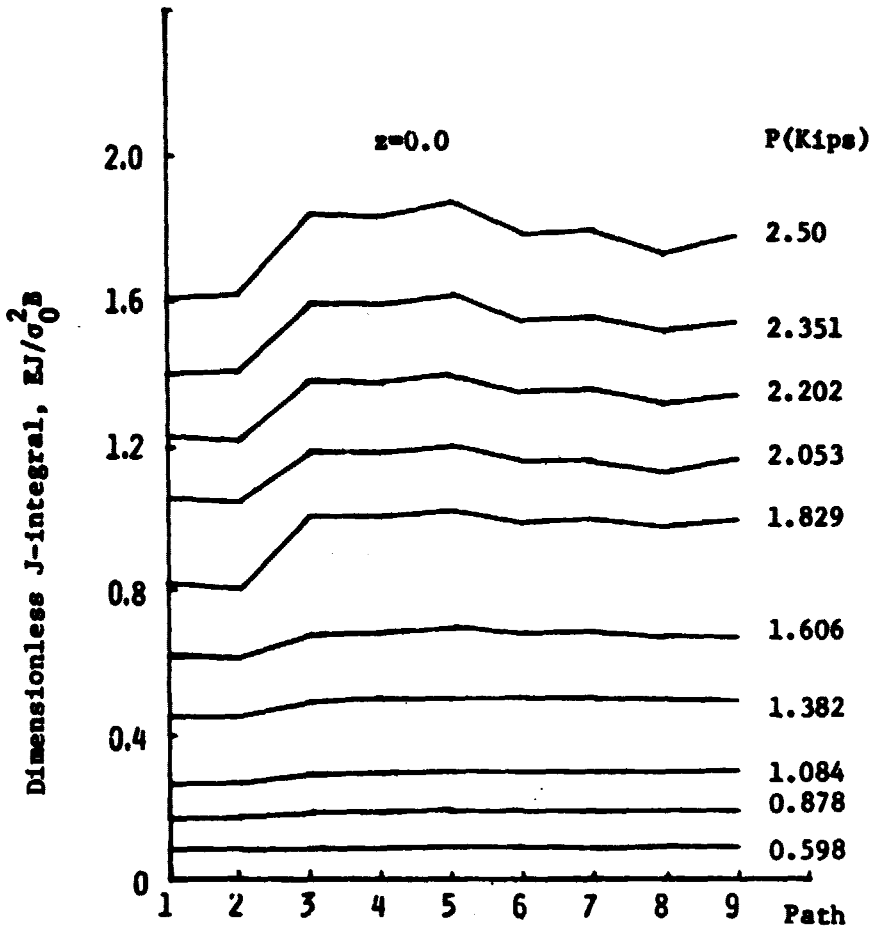

Figure 33. Non-dimensional J-integral values for different paths at the center of the CT specimen with curved crack front 1 for increasing load.

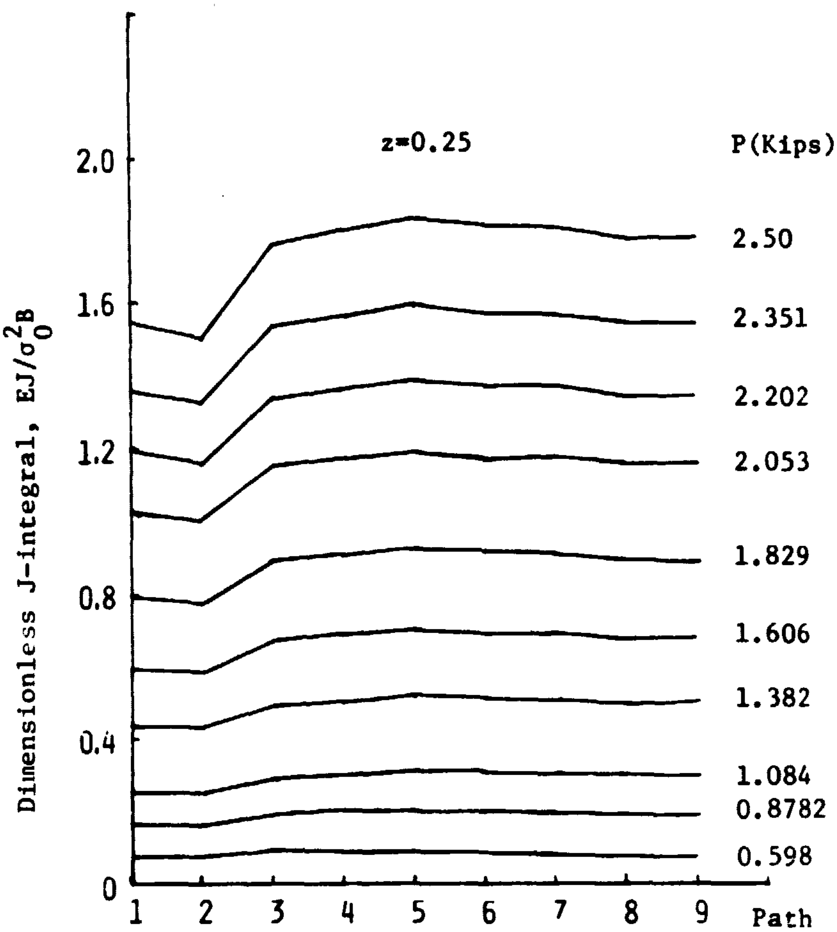

Figure 34. Non-dimensional J-integral values for different paths at midway between center and surface of the CT specimen with curved crack front 1 for increasing load.

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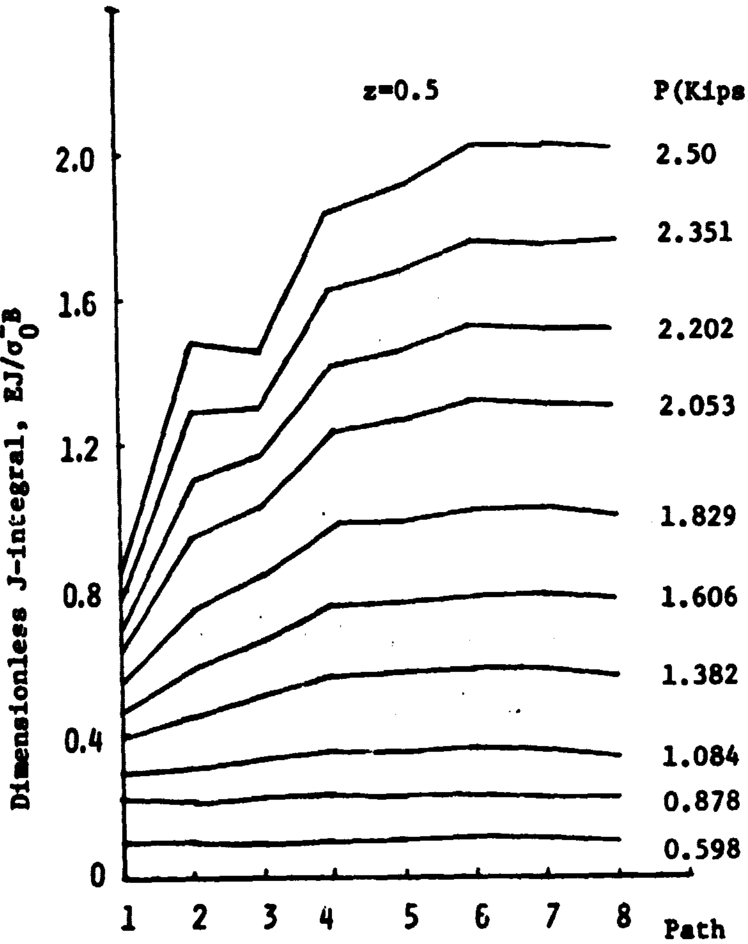

Figure 35. Non-dimensional J-integral values for different paths at the surface of the CT specimen with curved crack front 1 for increasing load.

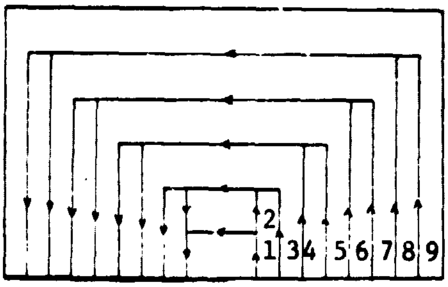

Figure 36(a). Different paths for the planes located at z=0.0 through 0.25

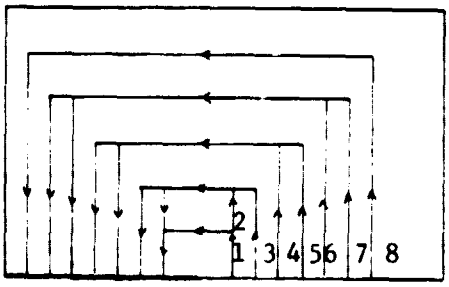

Figure 36(b). Different paths for the plane located at z=0.375

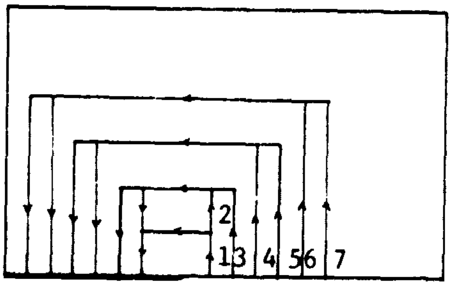

Figure 36(C). Different paths for the plane located at z=0.5

Figure 36. Different paths used for evaluating J-integral for curved crack front 2

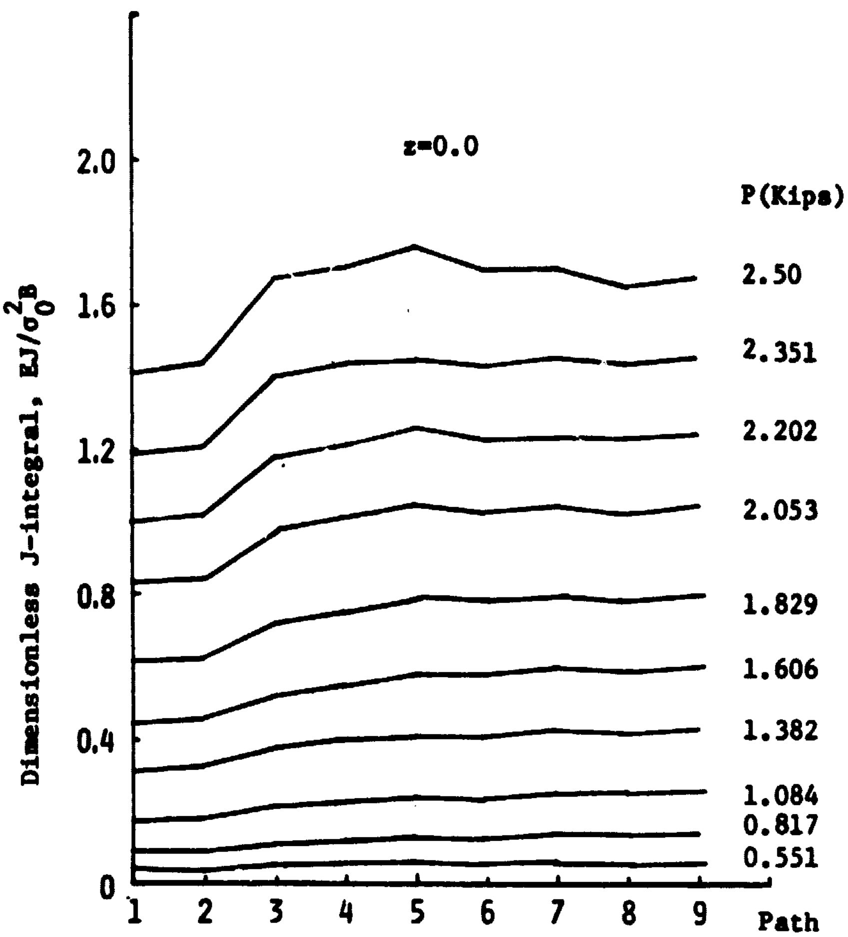

Figure 37. Non-dimensional J-integral values for different paths at the center of the CT specimen with curved crack front 2 for increasing load.

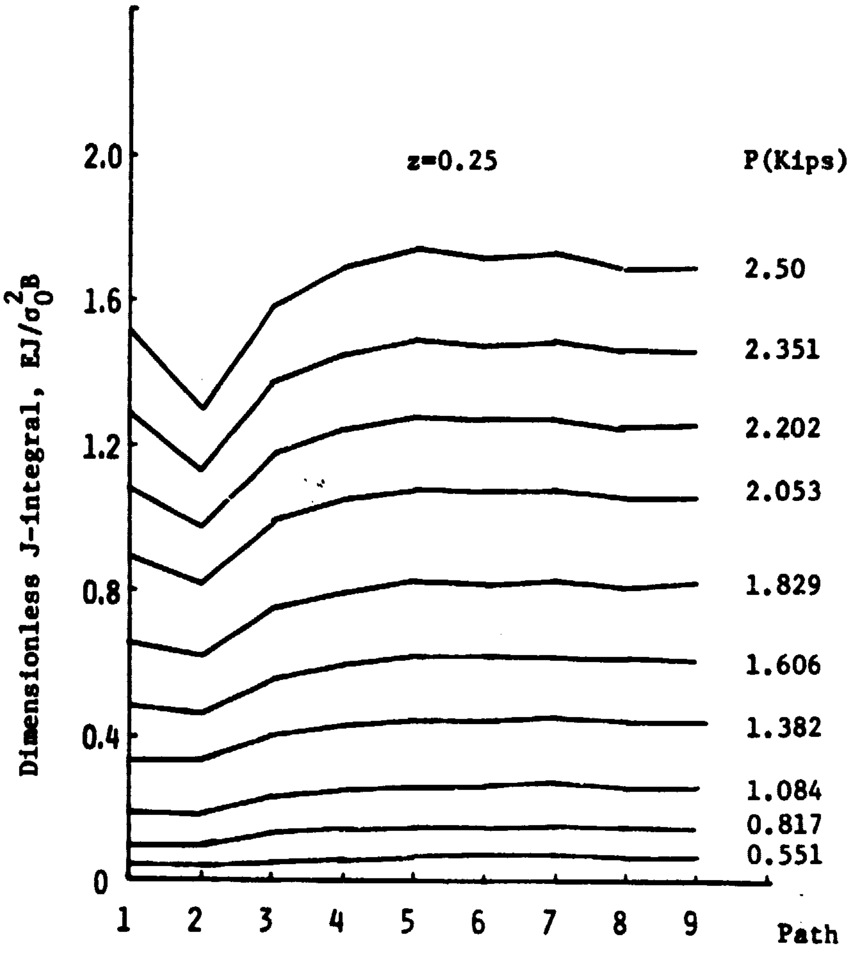

Figure 38. Non-dimensional J-integral values for different paths at midway between center and surface of the CT specimen with curved crack front 2 for increasing load.

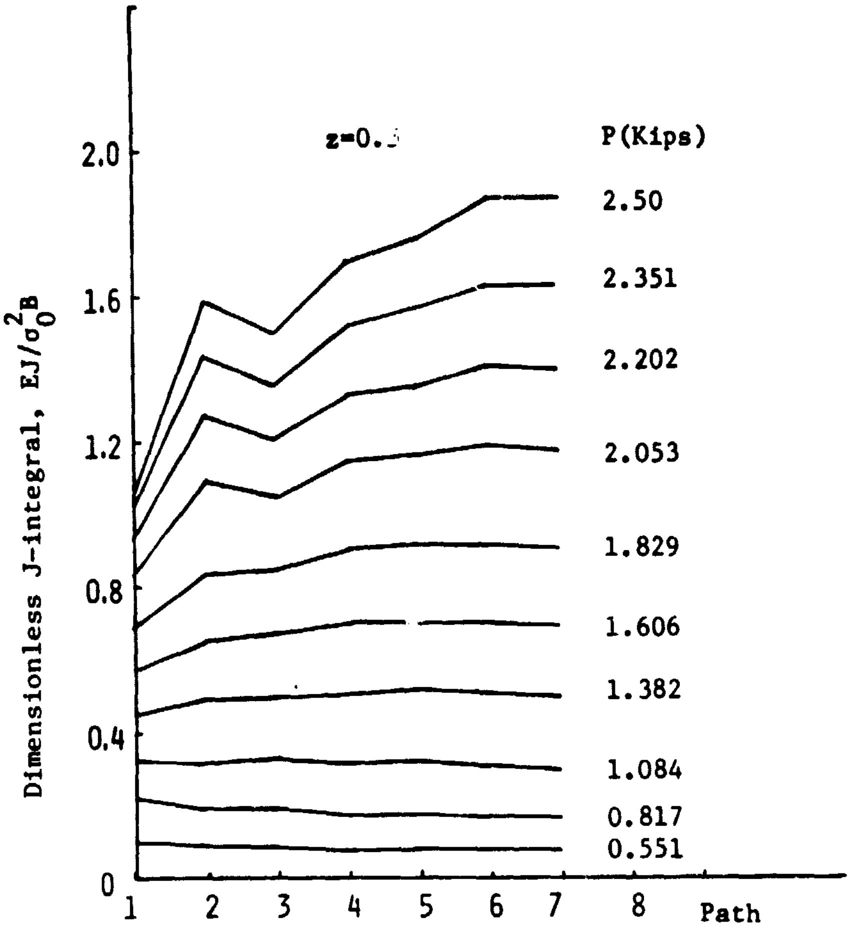

Figure 39. Non-dimensional J-integral values for different paths at the surface of the CT specimen with curved crack front 2 for increasing load.

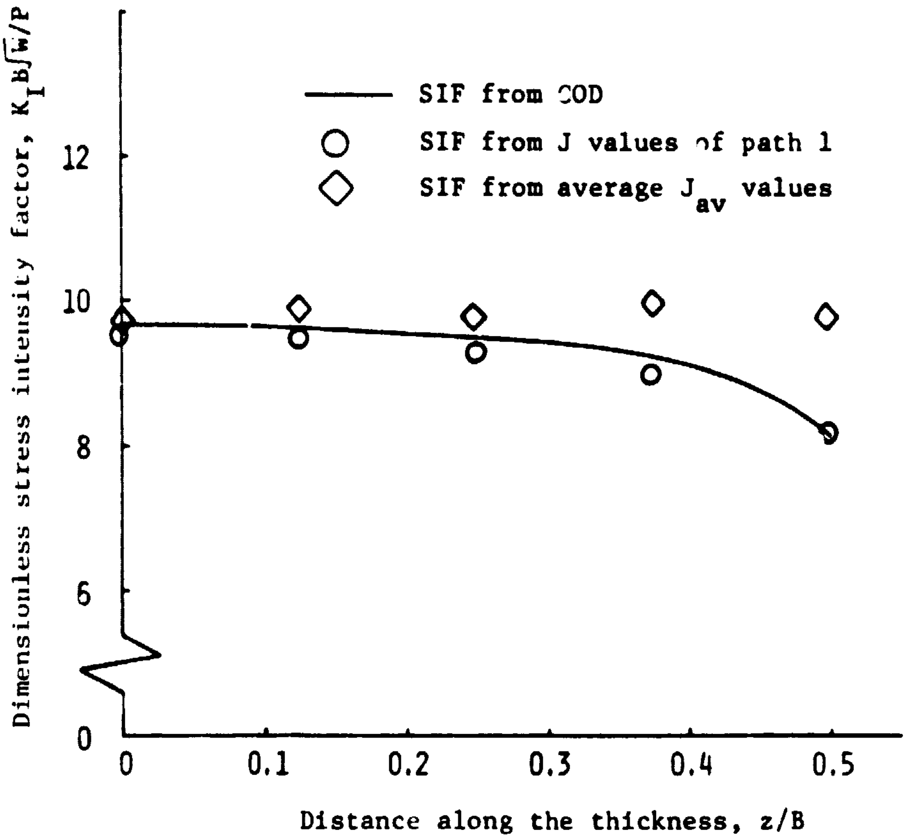

Figure 40. Variation of dimensionless stress intensity factor through the thickness of the compact tension specimen.

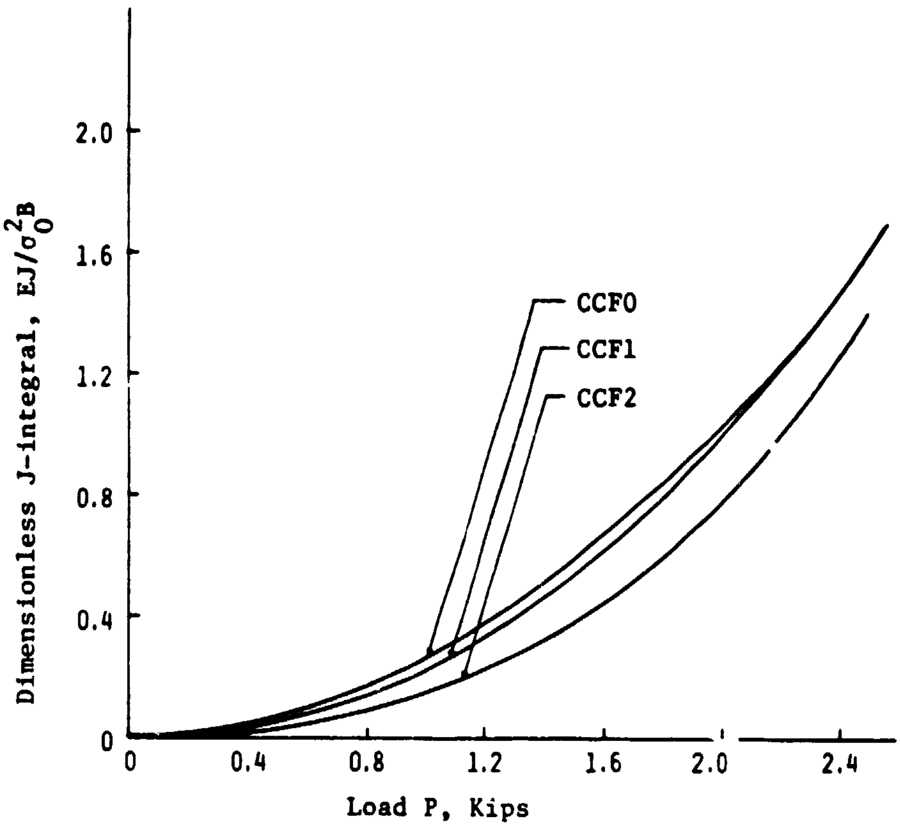

Figure 41. Non-dimensional values of J-integral at the center of the CT specimen for Path 1 versus the applied load for three different crack fronts, z=0.0

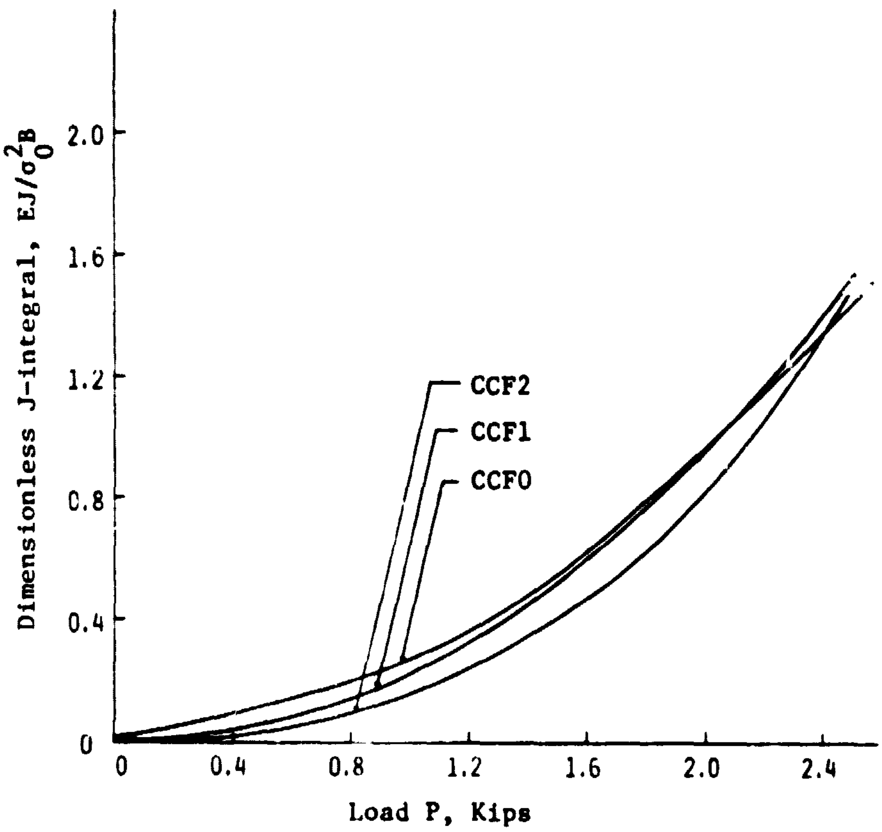

Figure 42. Non-dimensional values of J-integral at midway between center and surface of the CT specimen for Path 1 versus the applied load for three different crack fronts, z=0.25

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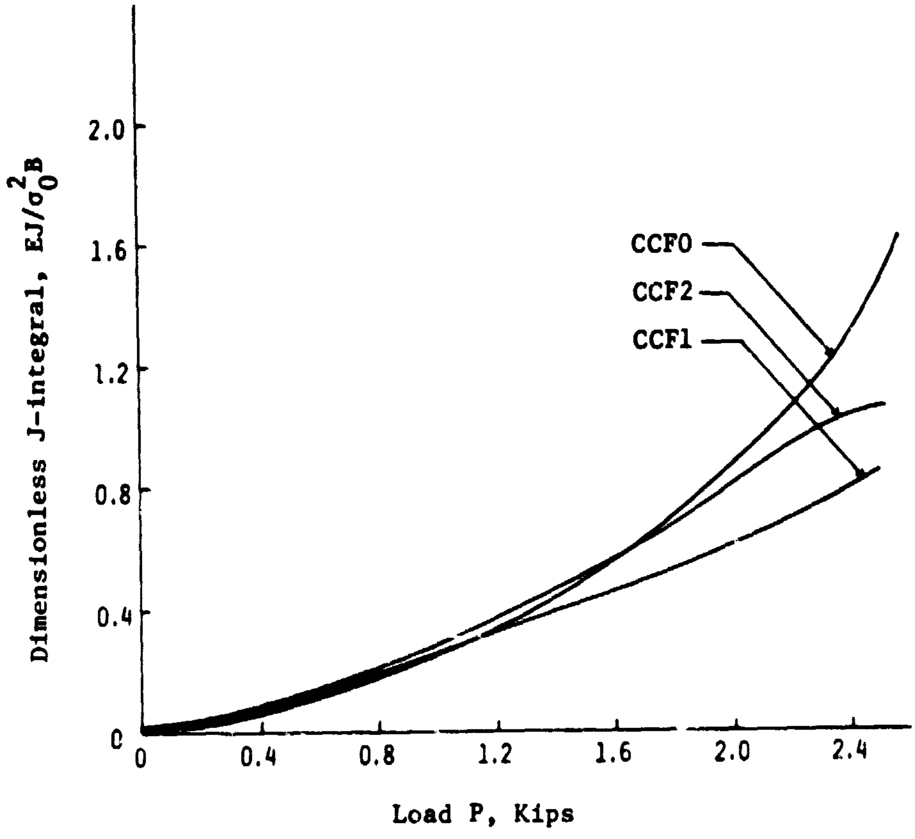

Figure 43. Non-dimensional values of J-integral at the surface of a CT specimen for Path 1 versus the applied load for three different crack fronts, z=0.5

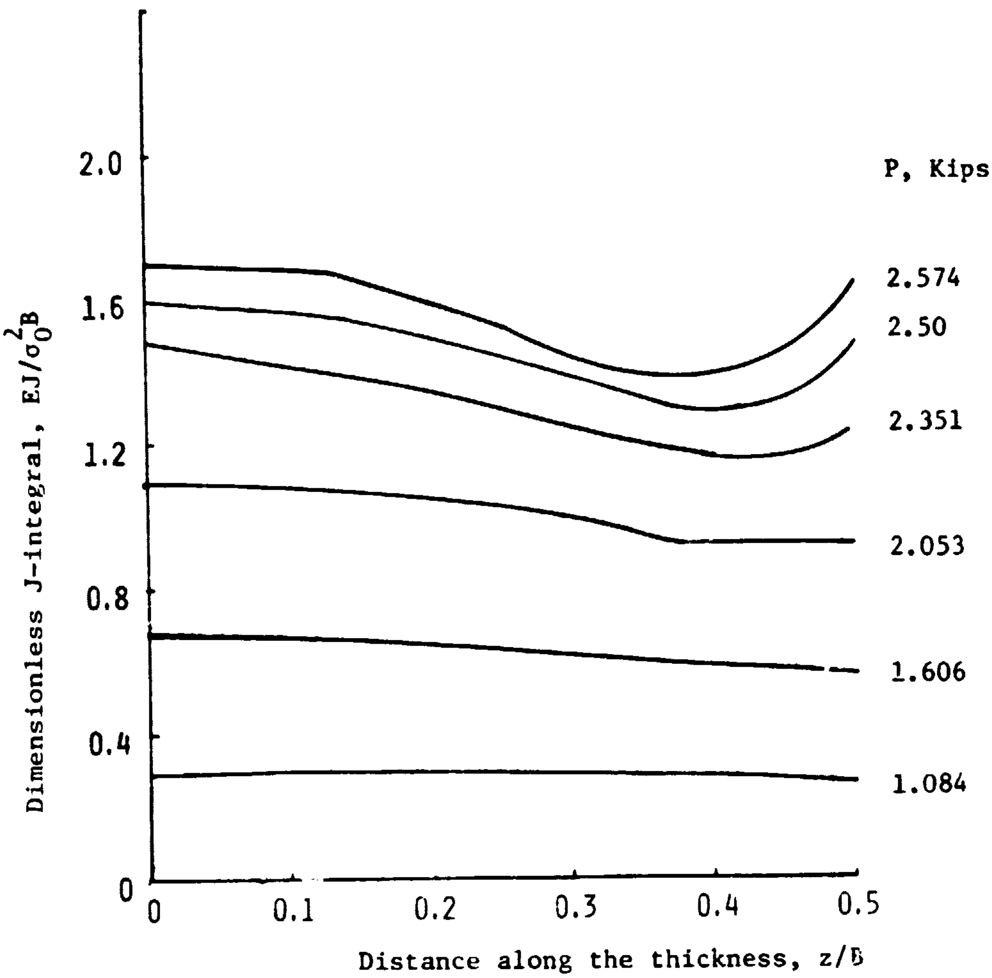

Figure 44. Variation of non-dimensional J-integral values through the thickness of a CT specimen with a straight crack with increasing appled load.

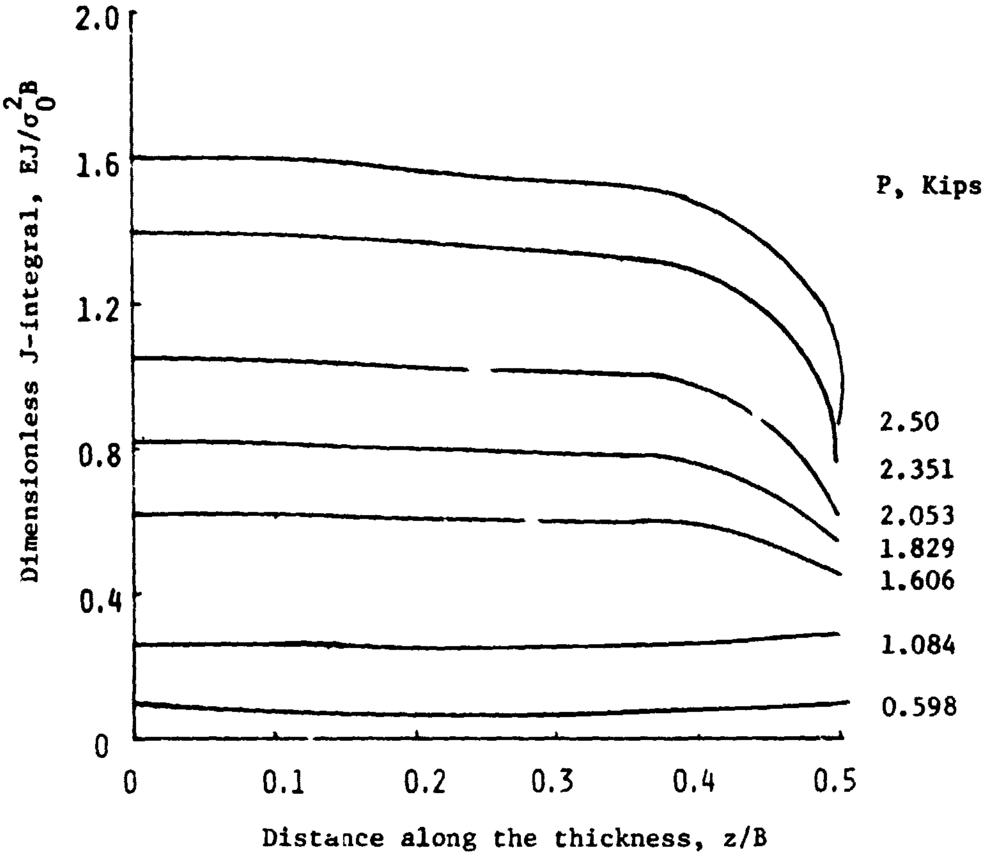

Figure 45. Variation of non-dimensional J-integral values through the thickness of the CT specimen with curved crack front 1 with increasing applied load.

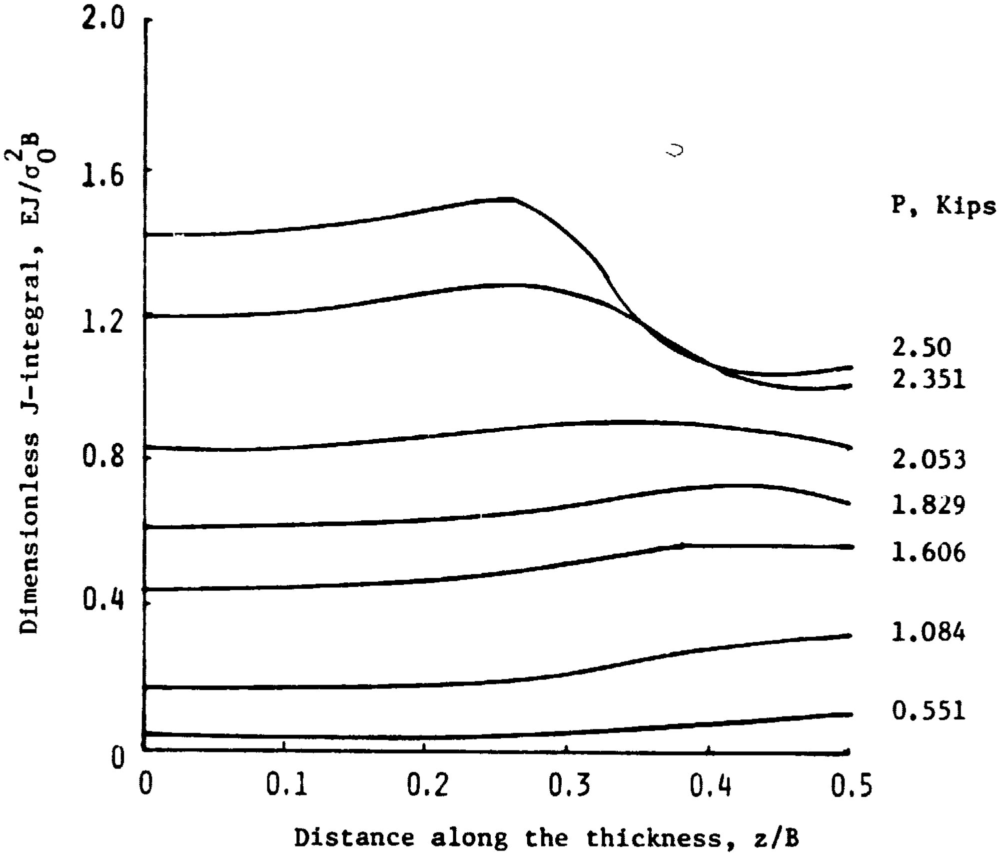

Figure 46. Variation of non-dimensional J-integral v[illegible]s through the thickness of the CT specim[illegible]h curved crack front 2 with increasing applied load.

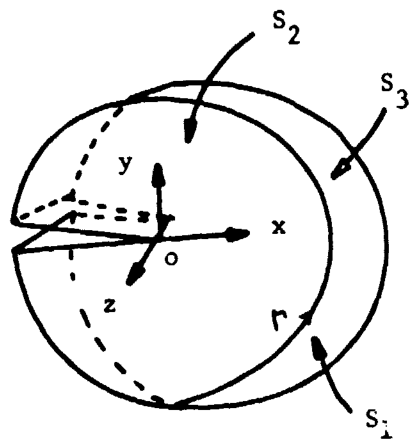

Figure 47. Section of a three-dimensional crack.

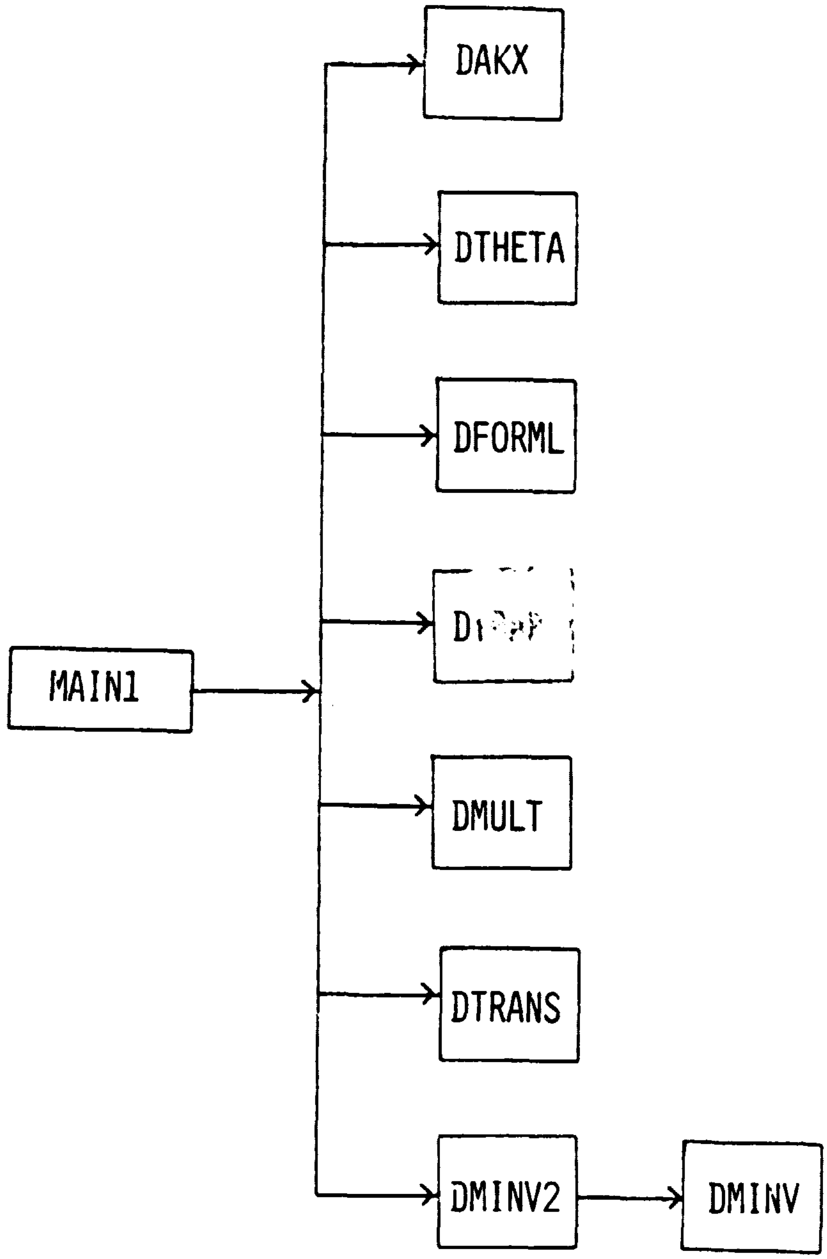

Figure 48. Schematic representation of computer program MAIN1

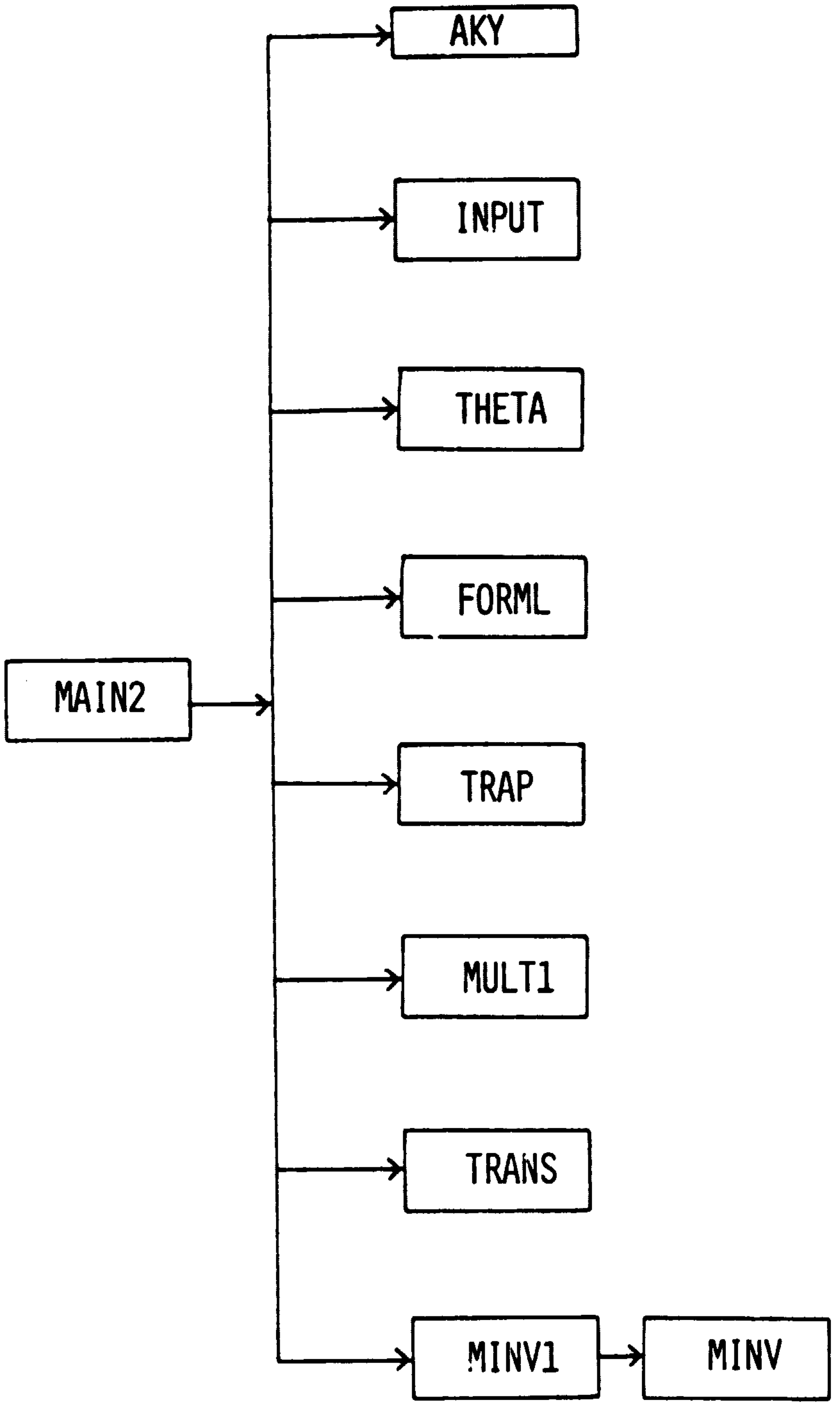

Figure 49. Schematic representation of compu*er program MAIN2

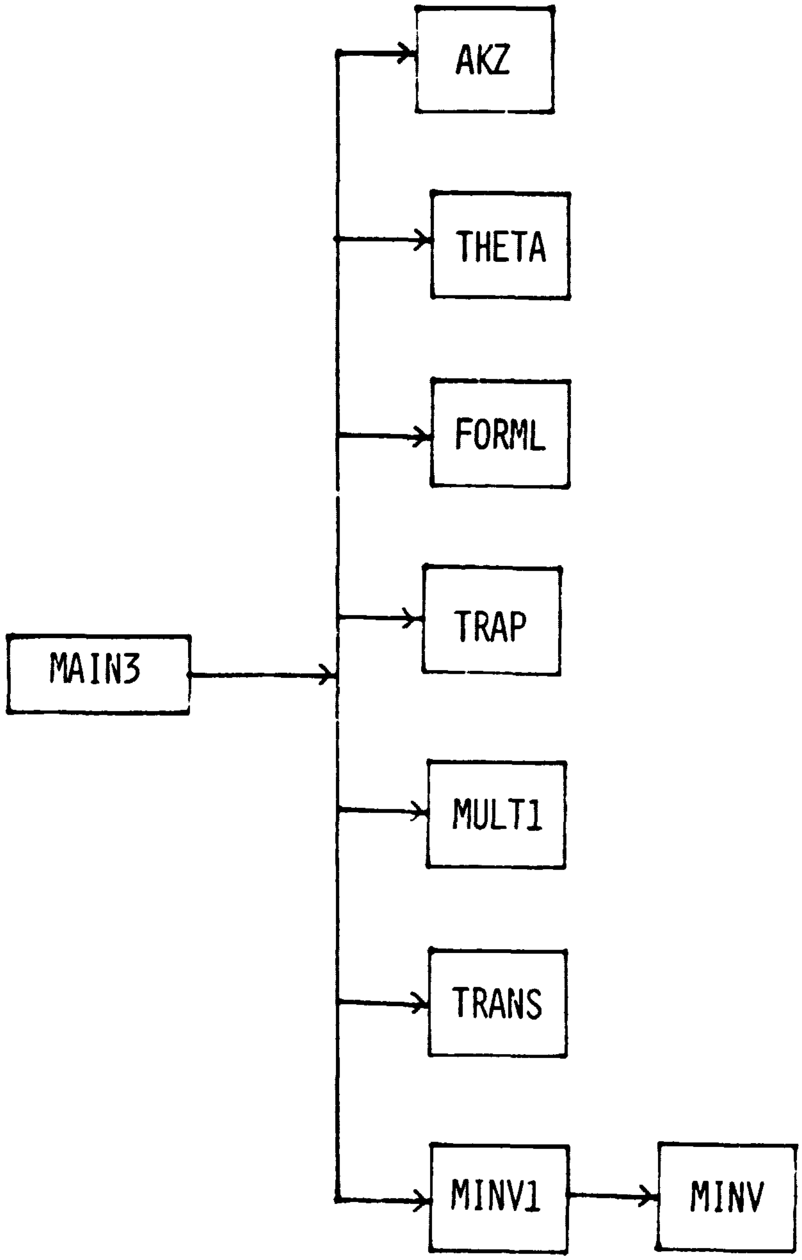

Figure 50. Schematic representation of computer program MAIN3

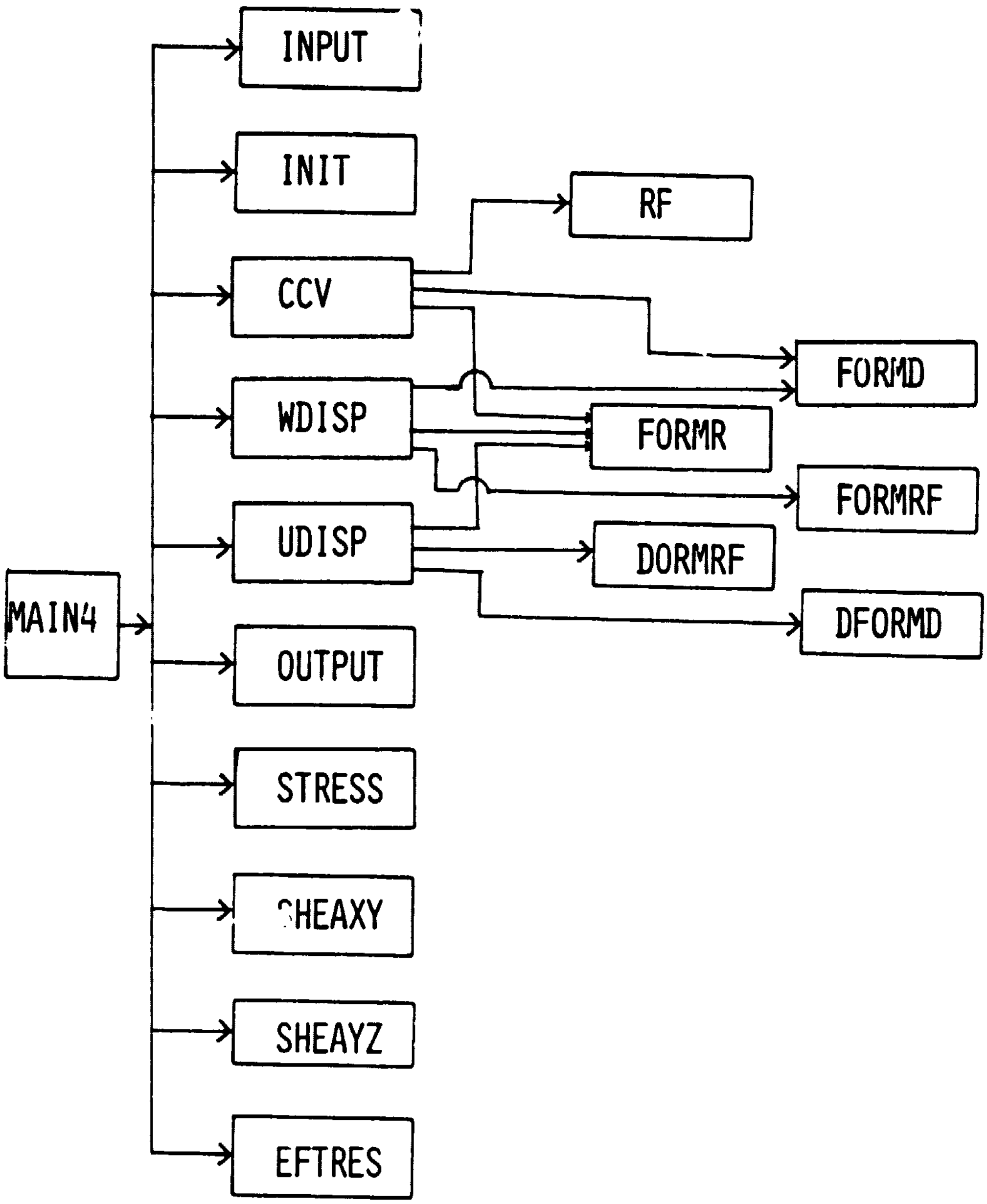

Figure 51. Schematic representation of computer program MAIN4

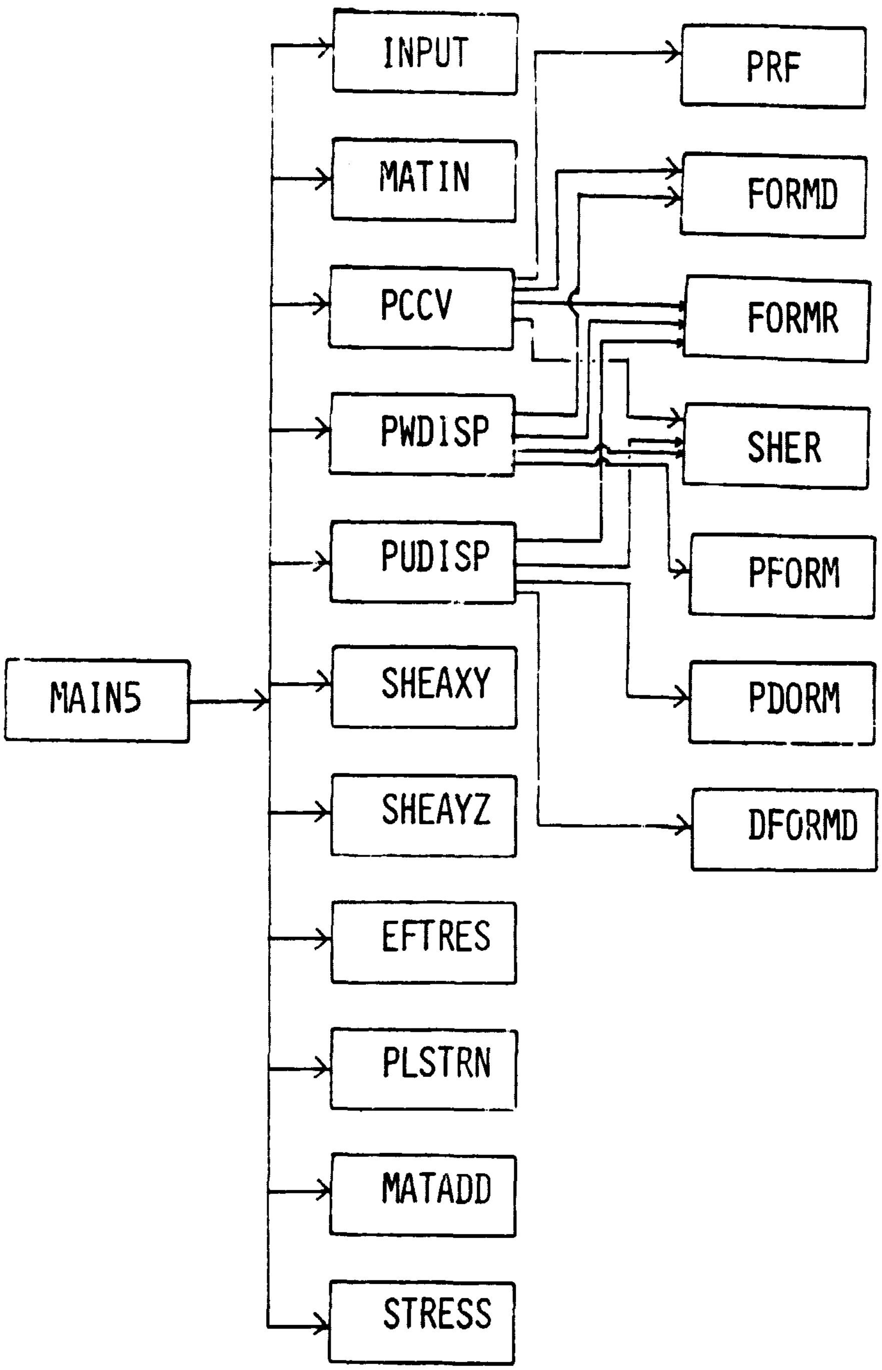

Figure 52. Schematic representation of computer program MAIN5

APPENDIX A

Formulation of the Governing Ordinary Differential Equations for the Lines Located at the Surface and Adjacent to the Surface of the Cracked Specimen

For boundary surface lines and lines adjacent to the boundary surface lines, the difference expressions for the second derivative will involve imaginary lines outside the boundary. Since three-dimensional elasticity problems have three boundary conditions at every point of the bounding surface and a second order ordinary differential equation needs only two conditions, the shear stress at the boundaries is used to eliminate the imaginary lines outside the surface while the condition of the prescribed normal traction or displacement will be enforced through the constants of the homogeneous solutions.

A.1 Derivation of Ordinary Differential Equation for x - Directional Lines

Let us take the x-directional line which is formed by the interaction of x - z and x - y coordinate planes of a solid (Figure 6). The shear stress for the free surface is given by

$$\sigma_{yx}\Big|_{x-z \text{ Coordinate plane}} = 0 \tag{A.1}$$

$$\text{or} \quad \partial u/\partial y + \partial v/\partial x = 0 \tag{A.2}$$

At line 1 using the imaginary displacement u_1^{fy} of the fictitious line 1^{fy} the equation (A.2) can be written as

$$\frac{u_2 - u_1^{fy}}{2\, h_y} = \frac{\partial v}{\partial x}\Big|_1 = \frac{dv}{dx}\Big|_1$$

or $$u_1^{fy} = u_2 + 2\, h_y\, (dv/dx)\Big|_1 \qquad \text{(A.3)}$$

The plane x-y is one of the symmetry planes and u displacements are symmetric with respect to this plane.

The fictitious displacement u_1^{fy} is used to write a forward finite difference equation for the partial derivative with respect to y, while two fictitious displacements u^{f1z} and u^{f2z} are used to write a central difference equation for the partial derivative with respect to z. The ordinary differential equation for line 1 is written as

$$\begin{aligned}&(d^2u_1/dx^2) + (C_1/12\, h_y^2)[11\, u_1^{fy} - 20\, u_1 + 6\, u_2 + \\ &4\, u_3 - u_4] + (C_1/12\, h_z^2)\,[-\, u_1^{f1z} + 16\, u_1^{f2z} \\ &-30\, u_1 + 16\, u_{NY+1} - u_{2\,NY+1}] = \\ &C_2\, \frac{d}{dx}\, (\dot{v} + \dot{w})\Big|_1 \end{aligned} \qquad \text{(A.4)}$$

Due to the symmetry considerations,

$$+\, u_1^{f1z} = u_{2\,NY+1}$$

and $$u_1^{f2z} = u_{NY+1} \qquad \text{(A.5)}$$

on substituting equations (A.3) and (A.5) into equation (A.4), we obtain

$$(d^2u_1/dx^2) +(C_1/12\ h_y^2) \quad [- 20\ u_1 + 17\ u_2 + 4\ u_3 - u_4]$$
$$+(C_1/12\ h_z^2) [- 30\ u_1 + 30\ u_{NY+1} - 2\ u_{2\ NY+1}]$$
$$= C_2 \frac{d}{dx} (\dot{v} + \dot{w})_1 -(11\ C_1/6\ h_y) \ (dv/dr)\Big|_1 \qquad (A.6)$$

Similarly for line 2, by making use of the fictitious line 1^{fy} and symmetry the ordinary differential equation can be written as

$$(d^2u_2/dx^2) + (C_1/12\ h_y^2) \quad [- u_1^{fy} + 16\ u_1 - 30\ u_2 +$$
$$16\ u_3 - u_4] \ +(C_1/12\ h_z^2) \ [- 30\ u_2 + 32\ u_{NY+2} -$$
$$2\ u_{2\ NY+2}] = C_2 \ \frac{d}{dx} \ (\dot{v} + \dot{w})_2 \qquad (A.7)$$

Note the use of the central difference equation for the partial derivative with respect to y. The fictitious displacement u_1^{fy} is once again eliminated by the use of equation (A.3), and the resulting ordinary differential equation for line 2 is written as follows

$$(d^2u_2/dx^2) + (C_1/12\ h_y^2) \quad [16\ u_1 - 31\ u_2 + 16\ u_3 - u_4]$$
$$+(C_1/12\ h_z^2) \ [- 30\ u_2 + 32\ u_{NY+2} - 2\ u_{2\ NY+2}]$$
$$= C_2 \frac{d}{dx} (\dot{v} + \dot{w})_2 +(C_1/6\ h_y)(dv/dx)\Big|_1 \qquad (A.8)$$

Bounding line 3 to NY - 2 do not require the use of any fic-

titious line; however the symmetry considerations will be used to derive the ordinary differential equations. For the line NY - 1 the ordinary differential equation can be written as

$$(d^2 u_{NY-1}/dx^2) + (C_1/12\,h_y^2)\,[-u_{NY-3} + 16\,u_{NY-2}$$

$$-30\,u_{NY-1} + 16\,u_{NY} - u_{NY}^{fy}] + (C_1/12\,h_z^2)\,[-30 \times$$

$$u_{NY-1} + 16\,u_{2\,NY-1} - 2\,u_{3\,NY-1}]$$

$$C_2\,\frac{d}{dx}(\dot{v} + \dot{w})_{NY-1} \qquad (A.9)$$

where u_{NY}^{fy} is the displacement at the fictitious line $_{NY}{}^{fy}$ and it is eleminated by making use of the shear stress boundary condition at that face

$$\sigma_{yx}\Big|_{face\ II} = 0 \qquad \partial v/\partial x + \partial u/\partial y = 0 \quad (A.10)$$

The equation (A.10) can be expanded as

$$\frac{u_{NY}^{fy} - u_{NY-1}}{2\,h_y} = -\left.\frac{dv}{dx}\right|_{NY}$$

$$\text{or} \quad u_{NY}^{fy} = u_{NY-1} - 2\,h_y\,(dv/dx)\Big|_{NY} \qquad (A.11)$$

The elimination of u_{NY}^{fy} from the equation (A.9) with the help of equation (A.11) results in the following ordinary differential equation for the line u_{NY-1}

$$(d^2 u_{NY-1}/dx^2) + (C_1/12\,h_y^2)\,[-u_{NY-3} + 16\,u_{NY-2}$$

$$-31\, u_{NY-1} + 16\, u_{NY}] + (C_1/12\, h_z^2) [-30\, u_{NY-1} +$$

$$16\, u_{2\ NY-1} - 2\, u_{3\ NY-1}] = C_2 \frac{d}{dx} (\dot{v} + \dot{w})_{NY-1}$$

$$-(C_1/6\, h_y)(dv/dx)\Big|_{NY} \qquad (A.12)$$

In the similar manner the ordinary differential equation can be written for the line NY, except that now a backward finite difference equation has to be used to write the difference equation for the partial derivative with respect to y. For example

$$(d^2u_{NY}/dx^2) + (C_1/12\, h_y^2) [-u_{NY-3} + 4\, u_{NY-2} +$$

$$6\, u_{NY-1} - 20\, u_{NY} + 11\, u_{NY}^{fy}] + (C_1/12 h_z^2) [-30 \times$$

$$u_{NY} + 32\, u_{2\ NY} - 2\, u_{3\ NY}] = C_2 \frac{d}{dx} (\dot{v} + \dot{w})_{NY}$$

$$(A.13)$$

Once again, the displacement u_{NY}^{fy} on the fictitious line NY^{fy} is eliminated with the help of equation (A.11) and it leads to following ordinary differential equation for the line NY,

$$(d^2u_{NY}/dx^2) + (C_1/12\, h_y^2) [-u_{NY-3} + 4\, u_{NY-2} + 17 \times$$

$$u_{NY-1} - 20\, u_{NY}] + (C_1/12 h_z^2) [-30\, u_{NY} + 32\, u_{2\ NY} -$$

$$2\, u_{3\ NY}] = C_2 \frac{d}{dx} (\dot{v} + \dot{w})_{NY} + (11/6\, h_y)(dv/dx)_{NY}$$

$$(A.14)$$

The ordinary differential equations for lines NY + 1 through ℓ - 2 NY can be written in the similar manner. For line 2 NY + 1 through ℓ - 2 NY there is no need to take symmetry into consideration to write finite difference equations for the partial differentials with respect to z, since there are enough points available in the grid to write central difference equations.

For the lines ℓ - 2 NY + 1 through ℓ, in addition to shearing stress σ_{xy} , the shearing stress σ_{zy} is also incorporated into the ordinary differential equations. This is done to eliminate the displacements appearing due to the use of fictitious lines while writing difference equations for the partial derivatives with respect to z. For example

$$\sigma_{zz}\big|_{\ell - NY + 1} = 0 \qquad \partial u/\partial z + \partial w/\partial x = 0 \qquad \text{(A.15)}$$

For the fictitious line ℓ - NY + 1, the above equation can be written as

$$\frac{u^{fz}_{\ell - NY + 1} - u_{\ell - 2NY + 1}}{2\,h_z} = \left.\frac{dw}{dx}\right|_{\ell - NY + 1}$$

or

$$u^{fz}_{\ell - NY + 1} = u_{\ell - 2\,NY + 1} - (2\,h_z)\left(\frac{dw}{dx}\right)\bigg|_{\ell - NY + 1}$$

(A.16)

Similarly by making use of the fictitious line $u^{fy}_{\ell - 2NY + 1}$, the shearing stress σ_{xy} is reduced to the following equation,

C-3

$$u^{fy}_{\ell-2NY+1} = u_{\ell-2NY+2} + 2\,h_y \left.\frac{dv}{dx}\right|_{\ell-2NY+1}$$

(A.17)

The ordinary differential equation for the line $\ell-2NY+1$ is written below,

$$(d^2u_{\ell-2NY+1}/dx^2)+(C_1/12h_y^2)\;[11\,u^{fy}_{\ell-2NY+1} - 20\,u_{\ell-2NY+1} + 6\,u_{\ell-2NY+2} + 4\,u_{\ell-2NY+3} - u_{\ell-2NY+4}] + (C_1/12\,h_z^2)[-u_{\ell-4NY+1} + 16\times u_{\ell-3NY+1} - 30\,u_{\ell-2NY+1} + 16\,u_{\ell-NY+1} - u^{fz}_{\ell-NY+1}] = C_2\,\frac{d}{dx}(\dot{v}+\dot{w})_{\ell-2NY+1} \qquad \text{(A.18)}$$

Elimination of $u^{fy}_{\ell-2NY+1}$ and $u^{fz}_{\ell-NY+1}$ from the equation (A.18) with the help of equations (A.16) and (A.17) results in the following ordinary differential equation,

$$(d^2u_{\ell-2NY+1}/dx^2)+(C_1/12h_y^2)\;[-20\,u_{\ell-2NY+1} + 17\,u_{\ell-2NY+2} + 4\,u_{\ell-2NY+3} - u_{\ell-2NY+4}] + (C_1/12h_z^2)[-u_{\ell-4NY+1} + 16\,u_{\ell-3NY+1} - 31\,u_{\ell-2NY+1} + 16\,u_{\ell-NY-1}] = C_2\,\frac{d}{dx}(\dot{v}+\dot{w})_{\ell-2NY+1} - (C_1/6\,h_y)(dv/dx)_{\ell-2NY+1} - (C_1/6\,h_z)\times (dw/dx)_{\ell-NY+1} \qquad \text{(A.19)}$$

The same method is used to construct the ordinary differential equations for the remaining lines. Appropriate difference equations are used to express the partial derivatives. For example for lines $\mathcal{L}$ - NY + 1 through $\mathcal{L}$, to write the difference equation with respect to z, a backward finite difference formula has to be used. Since there are not enough grid points to write central difference equations.

A.2 Derivation of Ordinary Differential Equations for y - Directional Lines

The first y-directional line is formed by the intersection of y-z and y-x coordinate planes (Figure 7). The shear stresses on these planes are utilized to eliminate the fictitious lines. The shear stress for the free surface y-z is

$$\sigma_{yx}\Big|\ \text{y-z coordinate plane} = 0 \qquad \text{(A.20)}$$

or

$$\partial u/\partial y + \partial u/\partial x = 0 \qquad \text{(A.21)}$$

Using the imaginary displacement v_1^{fx} of the fictitious line 1^{fx} the equation (A.21) is written as

$$\frac{v_{NZ+1} - v_1^{fx}}{2\,h_x} = -\frac{d}{dx}\Big|_1$$

or

$$v_1^{fx} = v_{NZ+1} + 2\,h_x(dv/dx)\Big|_1 \qquad \text{(A.22)}$$

The plane x-y is one of the symmetry planes and v - displacements are symmetric with respect to this plane. The fictitious displacement v_1^{fx} is used to write a forward finite difference equation for the partial derivative with respect to x. For the partial derivative with respect to z, a central difference equation is used by using two fictitious displacements v_1^{f1z} and

v_1^{f2z}. The ordinary differential equation for line 1 is written as

$$(d^2v_1/dy^2)+(C_1/12\ h_x^2)[\ 11\ v_1^{fx} - 20\ v_1 + 6\ v_{NZ+1} + 4\ v_{2\ NZ+1} - v_{3\ NZ+1}]+(C_1/12\ h_z^2)\ [-\ v_1^{f1z} + 16x\ v_1^{f2z} - 30\ v_1 + 16\ v_2 - v_3] = C_2\ \frac{d}{dy}\ (\dot{u} + \dot{w})_1 \quad (A.23)$$

Due to symmetry

$$v_1^{f1z} = v_3$$

and

$$v_1^{f2z} = v_2 \quad (A.24)$$

Substitution of equations (A.22) and (A.24) into the equation (A.23) leads to

$$(d^2v_1/dy^2)+(C_1/12\ h_x^2)[\ -\ 20\ v_1 + 17\ v_{NZ+1} + 4x\ v_{2\ NZ+1} - v_{3\ NZ+1}]+(C_1/12\ h_z^2)\ [-\ 30\ v_1 + 32\ v_2 - 2\ v_3\ ] = C_2\ \frac{d}{dy}\ (\dot{u} + \dot{w})_1 - (11\ C_1/6\ h_x)x\ (du/dy)\Big|_1 \quad (A.25)$$

In the similar manner by making use of fictitious line and the appropriate finite difference equations, the ordinary differential equation can be written for lines 2 through NZ - 2.

To write the equation for line NZ - 1, there is a need of fictitious line in the z direction. Let us denote the displacement on this fictitious line by v^{fz}_{NZ}. This displacement will be eliminated by making use of shear stress σ_{yz} on the free surface:

$$\sigma_{yz}\Big|_{\text{bounding plane III}} = 0$$

leads to

$$\partial v/\partial z + \partial v/\partial y = 0 \tag{A.26}$$

on expanding the equation (A.26) at line NZ, we have

$$\frac{v^{fz}_{NZ} - v_{NZ-1}}{2\, h_z} = -\frac{dw}{dy}\Big|_{NZ}$$

or

$$v^{fz}_{NZ} = v_{NZ-1} - 2\, h_z (\,dw/dy)\Big|_{NZ} \tag{A.27}$$

In the similar manner the displacement on the fictitious line $NZ - 1^{fx}$ is denoted by v^{fx}_{NZ-1}. Since

$$\sigma_{yx}\Big|_{\text{Bounding plane IV}} = 0$$

leads to $\partial u/\partial y + \partial v/\partial x = 0$ (A.28)

on expanding the equation (A.28) at line NZ - 1 leads to

$$\frac{v_{2\,NZ-1} - v^{fx}_{NZ-1}}{2\, h_x} = \frac{dv}{dx}\Big|_{NZ-1}$$

or

$$v^{fx}_{NZ-1} = v_{2\,NZ-1} + (2\,h_x)(dv/dx)\Big|_{NZ-1} \qquad (A.29)$$

on making use of these two fictitious lines the ordinary equation for the line NZ - 1 can be written as

$$(d^2v_{NZ-1}/dy^2) + (C_1/12h_x^2)\ [\,11\,v^{fx}_{NZ-1} - 20x$$

$$v_{NZ-1} + 6\,v_{2\,NZ-1} + 4\,v_{3\,NZ-1} - v_{4\,NZ-1}\,] +$$

$$(C_1/12\,h_z^2)\ [\,-\,v_{NZ-3} + 16\,v_{NZ-2} - 30\,v_{NZ-1} +$$

$$16\,v_{NZ} - v^{fz}_{NZ}\,] = C_2\,\frac{d}{dy}\,(\dot{u} + \dot{w})_{NZ-1} \qquad (A.30)$$

The fictitious displacements v^{fx}_{NZ-1} and v^{fz}_{NZ} are eliminated by substituting equations (A.28) and (A.29) into the equation (A.30) and the resulting equation is

$$(d^2v_{NZ-1}/dy^2) + (C_1/12h_x^2)\ [\,-\,20\,v_{NZ-1} + 17\,v_{2\,NZ-1}$$

$$+\,4\,v_{3\,NZ-1} - v_{4\,NZ-1}\,] + (C_1/12h_z^2)\ [\,-\,v_{NZ-3} +$$

$$16\,v_{NZ-2} - 31\,v_{NZ-1} + 16\,v_{NZ}\,] = C_1\,\frac{d}{dy}\,(\dot{u} + \dot{w})_{NZ-1}$$

$$-(11\,C_1/6\,hx)(du/dy)\Big|_{NZ-1} - (C_1/6\,h_x)(dw/dy)\Big|_{NZ}$$

$$(A.31)$$

Using the method described above, the ordinary differential equations for the lines NZ through m - 2 NZ can be written. An appropriate choice of fictitious line and the difference equation will always lead to a ordinary differential equation. From this

equation the displacement due to fictitious line is eliminated by using the shearing stress at the boundary surface. For example, for lines m - 2 NZ + 1 through m, we need to utilise shearing stress σ_{yx} on the free bounding plane I. In addition to this, for lines m - NZ - 1, m - NZ, m - 1 and m we need additional conditions which are provided by the shearing stress σ_{yz} on the free bounding surface on face III.

The ordinary differential equation for the line m can be written as

$$(d^2 v_m/dy^2) + (C_1/12\, h_x^2)[- v_{m-3NZ} + 4\, v_{m-2NZ} + 6\, v_{m-NZ} - 20\, v_m + 11\, v_m^{fx}] + (C_1/12\, h_y^2)[- v_{m-3} + 4\, v_{m-2} + 6\, v_{m-1} - 20\, v_m + 11\, v_m^{fz}] = C_2 x \frac{d}{dy}(\dot{u} + \dot{w})_m \quad \text{(A.32)}$$

Note the use of backward finite difference equations for both the partial derivatives. The displacements v_m^{fx} and v_m^{fz} on the fictitious lines are eliminated from the equation (A.32) by making use of the shearing stress at the free surface in the following manner,

$$\sigma_{yx}\Big|_{\text{plane I}} = 0$$

$$\partial u/\partial y + \partial v/\partial x = 0 \quad \text{(A.33)}$$

The equation (A.33) can be written as

$$\frac{v_m^{fx} - v_{m-NZ}}{2\,h_z} = -\left.\frac{du}{dy}\right|_m$$

or

$$v_m^{fx} = v_{m-NZ} - (2\,h_x)(du/dy)\Big|_m \tag{A.34}$$

Similarly

$$v_m^{fz} = v_{m-1} - (2\,h_z)(dw/dy)\Big|_m \tag{A.35}$$

With the help of equations (A.34) and (A.35) the two displacements which do not belong to grid displacements are eliminated and the new ordinary differential equation for the line m is

$$\begin{aligned}
&(d^2v_m/dx^2) + (C_1/12h_x^2)\,[-v_{m-3NZ} + 4\,v_{m-2NZ} + \\
&17\,v_{m-NZ} - 20\,v_m] + (C_1/12\,h_z^2)\,[-v_{m-3} + \\
&4\,v_{m-2} + 17\,v_{m-1} - 20\,v_m] = C_2\,\frac{d}{dy}\,(\dot{u} + \dot{w})_m + \\
&(11\,C_1/6\,h_x)(du/dy)\Big|_m + (11\,C_1/6\,h_z)(dw/dy)\Big|_m
\end{aligned} \tag{A.36}$$

A.3 Derivation of Ordinary Differential Equations for z - Directional Lines

As shown in Figure 8, the lines 1 through NXC are on the cracked face while lines NXC + 1 through NX are on the uncracked face. By making use of fictitious lines the ordinary differential equation for line 1 can be written as

$$(d^2w_1/dz^2)+(C_1/12\,h_y^2)\,[\,11\,w_1^{fy} - 20\,w_1 + 6\,w_{NX+1} + 4\,w_{2NX+1} - w_{3NX+1}] + (C_1/12\,h_x^2)\,[11\,w_1^{fx} - 20\,w_1 + 6\,w_2 + 4\,w_3 - w_4] = C_2\,\frac{d}{dz}\,(\dot{u} + \dot{v})_1 \qquad (A.37)$$

Where w_1^{fy} and w_1^{fx} are the displacements on the fictitious line 1^{fy} and 1^{fx}. The shear stress

$$\sigma_{xz}\Big|_{\text{plane IV}} = 0 \qquad (A.38)$$

leads to

$$\partial u/\partial z + \partial w/\partial x = 0 \qquad (A.39)$$

The above equation is expanded for the line 1 as follows,

$$\frac{w_2 - w_1^{fx}}{2\,h_x} \qquad \frac{du}{dz}\Big|_1$$

or

$$w_1^{fx} = w_2 + 2\,h_x\,(du/dz)\Big|_1 \qquad (A.40)$$

Similarly by making use of the shearing stress on face v(b)

we can write

$$w_1^{fy} = w_{NX+1} + 2\, h_y (dv/dz)\Big|_1 \qquad (A.41)$$

on substituting the equations (A.40) and (A.41) into the equation (A.37) leads to the following differential equation for the line 1

$$(d^2 w_1/dz^2) + (C_1/12\, h_y^2)\,[-20\, w_1 + 17\, w_{NX+1} + 4\, w_{2\,NX+1} - w_{3\,NX+1}] + (C_1/12\, h_x^2)\,[-20\, w_1 + 17\, w_2 + 4\, w_3 - w_4] = C_2 \frac{d}{dz}(\dot{u} + \dot{v})_1 - (11\, C_1/6\, h_x)(du/dz)_1 - (11\, C_1/6\, h_z)(dv/dz)\Big|_1 \qquad (A.42)$$

Similar procedure can be used to write the ordinary differential equations for the lines 1 through NXC. The w displacements are symmetric with respect to the plane v(a). This consideration helps in eliminating the displacements on fictitious lines. For example by making use of two fictitious lines ${}_{NX-1}^{f1y}$ and ${}_{NX-1}^{f2y}$ in y-direction the ordinary differential equation for the line NX - 1 can be written as follows

$$(d^2 w_{NX-1}/dz^2) + (C_1/12\, h_x^2)[-w_{NX-3} + 16\, w_{NX-2} - 30\, w_{NX-1} + 16\, w_{NX} - w_{NX}^{fx}] + (C_1/12\, h_y^2)[-w_{NX-1}^{f1y} + 16\, w_{NX-1}^{f2y} - 30\, w_{NX-1} + 16 w_{2NX-1} - w_{3NX-1}] = C_2 \frac{d}{dz}(\dot{u} + \dot{v})_{NX-1} \qquad (A.43)$$

Due to the symmetry

$$w^{f1y}_{NX-1} = w_{3\ NX-1} \qquad (A.44)$$

and

$$w^{f2y}_{NX-1} = w_{2\ NX-1}$$

The displacement w^{fx}_{NX} can be expressed as

$$w_{NX} = w_{NX-1} - 2\ h_x\ (du/dz)\Big|_{NX} \qquad (A.45)$$

With the help of the equations (A.44) and (A.45) the fictitious displacements w^{fx}_{NX} , w^{f1y}_{NX-1} and w^{f2y}_{NX-1} are eliminated from the equation (A.) and the resulting ordinary differential equation for line NX - 1 is

$$(d^2 w_{NX-1}/dz^2) + (C_1/12\ h_y^2)\ [-30\ w_{NX-1} + 32\ w_{2\ NX-1}$$

$$- 2\ w_{3\ NX-1}] + (C_1/12\ h_x^2)\ [-\ w_{NX-3} + 16\ w_{NX-2}$$

$$- 31\ w_{NX-1} + 16\ w_{NX}] = C_2\ \frac{d}{dz}\ (\dot{u} + \dot{w})_{NX-1} -$$

$$(C_1/6\ h_x)\ (du/dz)\Big|_{NX} \qquad (A.46)$$

Similar procedure is followed to write the ordinary differential equations for lines NX + 1 through n.

APPENDIX B

Application of the Recurrence Relation Method For y-Directional Lines

As shown in Figure 4, the presence of crack divides the face V into two faces, namely face V(a) and V(b), respectively. The face V(b) is a traction free surface. All the y-lines starting from this face will satisfy the following boundary condition.

$$\sigma_{yy}\Big|_{V(b)} = 0 \qquad \text{(B.1)}$$

On the face V(a), all the prescribed v-displacements are zero. Hence, the y-directional lines starting from this face will satisfy,

$$v\Big|_{V(a)} = 0 \qquad \text{(B.2)}$$

Using the stress - displacement relation equation (B.1) can be reduced to

$$(dv/dy)\Big|_{V(b)} = -\frac{\nu}{1-\nu}\,(du/dx + dw/dz)_{V(b)} \qquad \text{(B.3)}$$

The equations (B.2) and (B.3) can be assembled into a vector form by following the sequence of ordering of the y-direc-

tional lines. The assembled vectors are $V'_{1,k}$ and $V''_{1,k}$. The bector V at face V can be written as follows,

$$V_1 = \begin{bmatrix} V'_{1,u} \\ V'_{1,k} \\ V''_{1,k} \\ V''_{1,u} \end{bmatrix}_{2m\times 1} \tag{B.4}$$

where $V'_{1,u}$ and $V''_{1,u}$ represents the unknown part of the vector. To get the remaining boundary conditions the traction condition at face II is used. In the case of tensile loading we have an applied σ at the face, as shown in Figure 2. Using the stress-displacements relation, we can write

$$(dv/dy)\Big|_{II} = \frac{\sigma}{E}\left\{\frac{(1+\nu)\,(1-2\nu)}{(1-\nu)}\right\} - \frac{\nu}{1-\nu}\,(du/dx + dw/dz)_{II} \tag{B.5}$$

The equations (B.5) are again assembled into a vector form and are written as

$$V_{n,k} = \frac{\sigma}{E}\,\frac{(1+\nu)\,(1-2\nu)}{(1-\nu)} - \frac{\nu}{1-\nu}\,(\dot{u}+\dot{w})_{II} \tag{B.6}$$

The complete vector at face II can now be written as

$$V_n = \begin{bmatrix} V_{n,u} \\ V_{n,k} \end{bmatrix}_{2m\times 1} \tag{B.7}$$

Where, $V_{n,u}$ represents the unknown part of the vector. Following the recurrence relation method, an equation similar to the equation (3.96) is written below

$$V_n = D_n V_1 + F_n \tag{B.8}$$

The evalutation of matrix D_n and F_n is described in Chapter 2. For convenience matrix D_n and vector F_n is partitioned and the equation (B.8) can be rewritten as

$$\begin{bmatrix} V_{n,u} \\ V_{n,k} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_2 & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix} \begin{bmatrix} V'_{1,u} \\ V'_{1,k} \\ V''_{1,k} \\ V''_{1,u} \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \tag{B.9}$$

Where order of various matrices is

$V_{n,u}$	$m\text{x}1$	$V'_{1,u}$	$m'\text{x}1$	$V''_{1,k}$	$m\text{x}1$
$V_{n,k}$	$m\text{x}1$	$V'_{1,k}$	$(m-m')\text{x}1$	$V''_{1,u}$	$(m-m')\text{x}1$
β_{11}	$m\text{x}m'$	β_{13}	$m\text{x}m'$		
β_{21}	$m\text{x}m'$	B_{23}	$m\text{x}m'$		

$$\beta_{12} \quad mx(m-m') \quad \beta_{14} \quad mx(m-m')$$

$$\beta_{22} \quad mx(m-m') \quad \beta_{24} \quad mx(m-m')$$

where $m' = NZ \times NXC$

The equation (B.9) is expanded as,

$$V_{n,k} = \beta_{21} V'_{1,u} + \beta_{22} V'_{1,k} + \beta_{23} V''_{1,k} + \beta_{24} V''_{1,u} + \beta_2$$

The above equation could be rearranged as

$$V_{n,k} = \begin{bmatrix} \beta_{21} & \beta_{24} \end{bmatrix} \begin{bmatrix} V'_{1,u} \\ V''_{1,u} \end{bmatrix} + \begin{bmatrix} \beta_{22} & \beta_{23} \end{bmatrix} \begin{bmatrix} V'_{1,k} \\ V''_{1,k} \end{bmatrix} + \beta_2$$

or

$$\begin{bmatrix} V'_{1,u} \\ V''_{1,u} \end{bmatrix} = \begin{bmatrix} \beta_{21} & \beta_{24} \end{bmatrix}^{-1} \left[V_{n,k} - \begin{bmatrix} \beta_{22} & \beta_{23} \end{bmatrix} \begin{bmatrix} V'_{1,k} \\ V''_{1,k} \end{bmatrix} - \beta_2 \right] \qquad \text{(B.10)}$$

With the help of (B.10) all the unknowns in the vector V_1 can be determined. Once the complete vector V_1 is known, using the recurrence relation method displacement v and $\dot{v}$ are obtained

at all the grid points.

In the absence of normal stresses at face II of the specimen, the equation (B.5) needs to be modified by setting σ equal to zero. Another thing which deserves attention at this point is the adjustment of equation (B.9) for the situation in which curved crack front is present. In such cases the lines are divided in two sets. The first set satisfies the boundary condition given by the equation (B.1). Total number of these lines is given by NFREE. The second set of lines satisfy the equation (B.2) at its starting boundary and total number of such lines is given by NFIX. Once again the governing equation is similar to the equation (B.9) but now the vector V_1 can not be partitioned as shown in the equation (B.9). The distribution of the known and unknown elements of vector V_1 is dependent upon the way it is decided to fix particular grid point on face V, which in turn depends upon the shape of the curved crack front. To solve this new matrix equation, the vector V_1 is rearranged so that it could be written as in the equation (B.9). Now m' is no longer equal to NZ×NXC, but its new value is equal to NFREE. The change in the arrangement of V_1 vector requires a rearrangement of D_n matrix to maintain the same original equations. This is achieved by interchanging the columns in D_n matrix. For example, if v_i is interchanged by v_j in V_1 vector, then the column i in matrix D_n will be replaced by its j th column while in the place of column j the elements of column i will be placed. This adjustment gives rise

to a modified D_n and V_1 vector. The advantage of this procedure is that after this rearrangement, exactly the same procedure can be followed as explained before for the straight crack. Once the modified vector V_1' is known, with the help of information for each individual grid point on face v the vector V_1 is constructed. Finally it is used to obtain solutions for v and $\dot{v}$ for all the grid points using the equation (2.91) of the recurrence relation method.

APPENDIX C

Derivation Of The Differential Equations For y-Directional Lines For Shear Loading

The y-directional lines whose equations are directly affected by the shear loading are 1, 2, ...NZ and NZ + 1, NZ + 2, ...2 NZ. The shear stress σ_{yx} at face IV is

$$\sigma_{yx}\Big|_{IV} = -\tau(y) \tag{C.1}$$

where $\tau(y)$ is expressed as

$$\tau(y) = 4\tau_o \; (y/h) \; (1-y/h) \tag{C.2}$$

τ_o is the maximum value of shearing stress at $y = h/2$, and h is the semi length of the specimen as shown in Figure 1(b).

Using the stress-displacement relation, the equation (C.1) can be written as

$$\left\{(\partial u/\partial y) + (\partial v/\partial x)\right\}_{IV} = -\tau(y)/G \tag{C.3}$$

G is the shear modulus of the material. Using the procedure followed to reduce the second Navier - Cauchy equation to an ordinary differential equation, the modified equation for the line 1 is written as

$$(d^2v_1/dy^2) + (C_1/12\ h_x^2)[\ 11\ v_1^{fx} - 20\ v_1 + 6\ v_{NZ+1} - 4\ v_{2NZ+1} - v_{3NZ+1}] + (C_1/12\ h_z^2)[-\ v_1^{f1z} + 16\ v_1^{f2z} - 30\ v_1 + 16\ v_2 - v_3] = C_2\ \frac{d}{dy}(\dot{u} + \dot{w})_1 \qquad (C.4)$$

Due to symmetry

$$v_1^{f1z} = v_3$$

and

$$v_1^{f2z} = v_2 \qquad (C.5)$$

The equation (C.3) can be expanded for line 1 in the following manner,

$$\frac{v_1^{fx} + v_{NZ+1}}{2\ h_x} = -\left.\frac{du}{dy}\right|_1 - \frac{\tau(y)}{G}$$

or

$$v_1^{fx} = v_{NZ+1} + (2\ h_x)[\ (du/dy)_1 + (\tau(y)/G)] \qquad (C.6)$$

Elimination of fictitious lines v_1^{fx}, v_1^{f1z} and v_1^{f2z} from equation (C.4) with the help of equations (C.5) and (C.6) leads to

$$(d^2v_1/dy^2) + (C_1/12\ h_x^2)[-\ 20\ v_1 + 17\ v_{NZ+1} + 4\ v_{2NZ+1} - v_{3NZ+1}] + (C_1/12\ h_z^2)[-\ 30\ v_1$$

$$32\ v_2 - 2\ v_3] = C_2\ \frac{d}{dy}\ (\dot{u} + \dot{w})_1\ - 11(C_1/6\ h_x)x$$

$$(du/dy)_1 - (11\ C_1/6\ h_x)(\tau(y)/G) \qquad (C.7)$$

In a similar manner equations for lines 2 through NZ can be modified. A comparison of equation (C.7) with the equation (2.30) shows that the two equations are nearly the same except the additional term of $-(11\ C_1/6\ h_x)(\tau(y)/G)$ which appears due to a non-zero prescribed shearing stress σ_{xy}. The modified equations for line 2 though NZ also include the additional term $-(11\ C_1/6\ h_x)$ $(\tau(y)/G)$. Other terms for these equations remain the same as before. The equation for line NZ + 1 is written as

$$(d^2 v_{NZ+1}/dy^2) + (C_1/12\ h_z^2)[-\ v_{NZ+1}^{f1z} + 16\ v_{NZ+1}^{f2z}$$

$$- 30\ v_{NZ+1} + 16\ v_{NZ+2} - v_{NZ+3}] + (C_1/12\ h_x^2)x$$

$$- v_1^{fx} + 16\ v_1 - 30\ v_{NZ+1} + 16\ v_{2NZ+1} - v_{3NZ+1}]$$

$$= C_2\ \frac{d}{dy}\ (\dot{u} + \dot{w})_{NZ+1} \qquad . \ (C.8)$$

Due to symmetry of v displacements

$$v_{NZ+1}^{f2z} = v_{NZ+2}$$

$$v_{NZ+1}^{f1z} = v_{NZ+3} \qquad (C.9)$$

On substituting the equations (C.6) and (C.9) into the equa-

tion (C.8) leads to the following ordinary differential equation for the y-directional line NZ + 1,

$$
\begin{aligned}
&(d^2 v_{NZ+1}/dy^2) + (C_1/12\ h_z^2)[- 30\ v_{NZ+1} + \\
&32\ v_{NZ+2} - 2\ v_{NZ+3}] + (C_1/12\ h_x^2)[16\ v_1 - \\
&31\ v_{NZ+1} + 16\ v_{2NZ+1} - v_{3NZ+1}] \\
&= C_2 \frac{d}{dy} (u + w)_{NZ+1} + (C_1/6h_x)(du/dy)_1 + (C_1/6h_x)(\tau(y)/G)
\end{aligned}
\tag{C.10}
$$

Once again the difference between equation (C.10) and the one used in section 2.2.2 is only the additional term $(C_1/6\ h_x)$x $(\tau(y)/G)$,appearing due to the prescription of a non-zero shearing stress. The equations for lines NZ + 2 through 2NZ will also have this additional term of $(C_1/6\ h_x)\ (\tau(y)/G)$.

Introducing the matrix notation, all the ordinary differential equations along the y-directional lines are expressed in the form

$$
\underset{m\times 1}{\frac{d^2}{dy^2} \left\{ v \right\}} = \underset{m\times m}{\left(K_y \right)} \underset{m\times 1}{\left\{ v \right\}} + \frac{d}{dy} \underset{m\times 1}{\left\{ s\ (y) \right\}} + \underset{m\times 1}{\left\{ s^*(y) \right\}}
\tag{C.11}
$$

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APPENDIX D

Three Dimensional J-Integral

The definition of J-integral presented in this appendix is taken from reference (37). Consider the section of three dimensional crack front shown in Figure 47. The surfaces, S_1, S_2 and S_3 surrounds the crack front. The component J_x of the general J-integral vector $\vec{J}$ is given as

$$J_x = \int_\Gamma (W dy - \sigma_{ij} n_j u_{i,x} ds_j)$$

$$\int_{S2 + S3} (\sigma_{iz} u_{i,x})/_{,z} \quad dS \qquad \text{(D.1)}$$

where i = 1, 2, 3 and Γ is a circular path as shown in Figure 47. W is strain energy density, σ_{ij} is stress tensor and u_i is referred to as displacement vector.

In two dimensional plane stress problems σ_{iz} becomes zero and the equation for J_x becomes the same as given by Rice (27). In the case of plane strain problem σ_{zz} is not zero but its variation through the thickness is zero. Therefore, the surface integral term drops from the equation (D.1) and once again the equation for J_x becomes the same as given by Rice (27).

APPENDIX E

Description And Listing Of The Computer Programs

In this appendix a description of the computer programs used in the present study is given. A complete listing of all these programs including subprograms is attached at the back of the thesis. In writing the various programs a strict modular approach was followed. This has helped in writing many subprograms which are commonly shared by several main programs.

The program was divided into five main programs. This was done to divide one complete run into several small runs. This was done to safeguard against the failure of computer system . The schematic representation of the computer program MAIN1 is shown in Figure 48. This program is employed to evaluate the various matrices needed for the x-directional calculations. At the end of the computer program run, all these matrices are stored on the virtual disc storage. Computer programs MAIN2 and MAIN3 whose schematic representation are given in Figures 49-50, are also used to calculate different matrices used for y and z-directional calculations. The outputs from these programs are also saved on the virtual disc storage.

A dynamic storage allocation scheme was used for all the

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programs. Consequently whenever there was a change in the number of grid lines, only the DIMENSION statements in the main programs, need to be modified. Due to the serious round off error problem, all the calculations for the x-direction were performed in double-double precision (128 bit word size). For the other two directions the double precision was sufficient.

The computer program MAIN4 is used to obtain the elastic solution for different cases. The subroutines UDISP, CCV, and WDISP perform the calculation for x, y and z-directional lines, respectively. In the case of tensile loading subroutine CCV was replaced by subroutine CVDISP. The subroutine OUTPUT is used to print the results for the displacements and their derivatives. The normal and shear stresses are calculated with the help of subroutines STRESS, SHEAXY, SHEAYZ. The effective stress at each grid point can be computed by using subroutine EFTRES. The output from the program MAIN4 was also stored on the virtual disc.

Four virtual tapes are needed to run the program MAIN4. The stored matrices which were originally calculated by programs MAIN1, MAIN2 and MAIN3 are read into three sequential virtual tapes. The fourth virtual tape is needed to write the output of program MAIN4 on the virtual disc.

Program MAIN5 is used for the elastic-plastic stress analysis. Its schematic representation is shown in Figure 52. There are many subroutines which are shared by this program with the program MAIN4. Due to the addition of plastic strain terms many

old subroutines were modified. Note the presence of new subprograms PUDISP, PCCV and PWDISP, used to perform calculations for the x, y, and z-directional lines, respectively. This program uses five virtual tapes. Three tapes are utilized to read the different matrices generated by the programs MAIN1, MAIN2, and MAIN3. The remaining other two virtual tapes are used to read and write the solution, at the beginning and at the end of the increment.

This format of programming provided considerable flexibility and a very good protection in case of computer systems failure during the execution of the program. In case of a system failure, only the solution for the current increment is lost, which can be recomputed by using the stored solution for the previous increment.

```
C.......................................................
C.                                                     .
C.                     PROGRAM MAIN1                   .
C.    PROGRAM TO FORM XK,XK1,XLOC,XLOC1,XLL,XLN MATRICES .
C.                                                     .
C.......................................................
C
        IMPLICIT REAL*16(A-H,O-Z)
C
      DIMENSION XK(35,35),XK1(35,35),XLOC(35,35),XLOC1(35,35)
      DIMENSION XLL(70,70),XLN(70,70),L(70),M(35),TEMP(70)
C
C
C      INPUT TO THE PROGRAM
C
        READ(5,200) NITER,NEXEC
        WRITE(6,1003) NITER
 1003   FORMAT(5X,'***** NO. OF ITERATION ALLOWED *****',I5//)
       READ(5,100)SIGMA,EPS
       READ(5,100)ALEN,WID,THIC
 100   FORMAT(3F10.0)
       READ(5,200)NX,NY,N7,NXC
       READ(5,200)NMX,NMY,NMZ
 200   FORMAT(4I5)
C
C      CALCULATION OF VARIOUS CONSTANTS
C
       PR=1./3.
       C3=1-PR
       C1=0.5*(1-2.*PR)/C3
       C2=-0.5/C3
       C3=-PR/C3
       C4=(1+PR)*(1-2.*PR)/(1-PR)
       HX=WID/(NX-1)
```

```
      HY=ALEN/(NY-1)
      HZ=THIC/(NZ-1)
      HMX=HX/NMX
      HMY=HY/NMY
      HMZ=HZ/NMZ
      WRITE(6,1001)NX,NY,NZ,NXC
      WRITE(6,1001)NMX,NMY,NMZ
      WRITE(6,1000)ALEN,WID,THIC,EPS
      WRITE(6,1000)C1,C2,C3,C4
      WRITE(6,1000)HX,HY,HZ
      WRITE(6,1000)HMX,HMY,HMZ
 1001         FORMAT(1X,5I5)
 1000         FORMAT(10F10.4)
      LX=NY*NZ
      LX1=2*LX
      LY=NZ*NX
      LY1=2*LY
      LZ=NX*NY
      LZ1=2*LZ
C
C
C     X-DIRECTION SOLUTION
C
C     FORMULATION OF KX AND THETA MATRIX
C
      CALL DAKX(XK,LX,NY,NZ,HY,HZ,C1,C3)
C
C
      CALL DTHETA(XK,LX,HMX,XK1)
      CALL DMINV(XK1,LX,D,L,M)
C
C     XK1 CONTAINS THETA INVERSE FOR KX MATRIX
C
      WRITE(6,5000)D
```

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```
      CALL DFORML(XK,XK1,LX,LX1,HMX,XLL)
      WRITE(6,5003)
 5003 FORMAT(1X,'THE LX MATRIX IS'/)
       CALL DTRAP(XLL,XLN,LX1,LX1)
      K=(NX-1)*HMX - 1
      DO 10 I=1,K
       CALL DMULT(XLN,XLL,LX1,TEMP)
         WRITE(6,1001)I
 10   CONTINUE
       IWRK=LX1
       MN=LX1*LX1
       CALL DTRANS(XLN,LX1,LX1,MN,L,IWRK,IOK)
       WRITE(6,1001)IOK
      WRITE(6,5005)
 5005 FORMAT(1X,'THE DNX MATRIX IS'/)
C
C      CALCULATION OF VARIOUS ALPHAS
C
C        INTERCHANGE OF LAST ROW OF XLN WITH LXTH ROW
C
         DO 510 J=1,LX1
         XLN(LX1,J)=XLN(LX,J)
 510      CONTINUE
C
         DO 340 I=1,LX
          II=LX + I
          DO 340 J=1,LX
       JJ=LX + J
       XLOC1(I,J)=XLN(II,JJ)
           XLOC(I,J)=XLN(II,J)
 340      CONTINUE
      CALL DMINV2(XLOC,LX,L,M)
C      IN SITU TRANSPOSITION OF XLL,YLL,ZLL
C
```

```
      IWRK=LX1
      MN=LX1*LX1
      CALL DTRANS(XLL,LX1,LX1,MN,L,IWRK,IOK)
      WRITE(6,1001)IOK
      REWIND 1
     WRITE(1)((XK1(I,J),I=1,LX),J=1,LX),((XLL(I,J),I=1,LX1),J=1,LX1),
    * ((XLOC1(I,J),I=1,LX),J=1,LX),((XLOC(I,J),I=1,LX),J=1,LX)
     STOP
5000  FORMAT(1X,'THE DETRMINANT IS',3X,E12.4)
    END
```

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```
C.............................................................
C.                                                           .
C.                      PROGRAM MAIN2                        .
C.              PROGRAM TO FORM YK,YK1,BK,YLL,YLN            .
C.                                                           .
C.............................................................
C
        IMPLICIT REAL*8(A-H,O-Z)
C
      DIMENSION YK(100,100),YK1(100,100),BK(100,100),YLL(200,200)
      DIMENSION YLN(200,200),L(200),M(100),TEMP(200),B23(100,50)
      DIMENSION ID(100)
C
C
C       INPUT TO THE PROGRAM
C
        READ(5,200) NITER,NEXEC
        WRITE(6,1003) NITER
 1003   FORMAT(5X,'***** NO. OF ITERATION ALLOWED *****',I5//)
       READ(5,100)SIGMA,EPS
       READ(5,100)ALEN,WID,THIC
 100   FORMAT(3F10.0)
       READ(5,200)NX,NY,NZ,NXC
       READ(5,200)NMX,NMY,NMZ
 200   FORMAT(4I5)
C
C      CALCULATION OF VARIOUS CONSTANTS
C
       PR=1./3.
       C3=1-PR
       C1=0.5*(1-2.*PR)/C3
       C2=-0.5/C3
       C3=-PR/C3
       C4=(1+PR)*(1-2.*PR)/(1-PR)
```

```
      HX=WID/(NX-1)
      HY=ALEN/(NY-1)
      HZ=THIC/(NZ-1)
      HMX=HX/NMX
      HMY=HY/NMY
      HMZ=HZ/NMZ
      WRITE(6,1001)NX,NY,NZ,NXC
      WRITE(6,1001)NMX,NMY,NMZ
      WRITE(6,1000)ALEN,WID,THIC,EPS
      WRITE(6,1000)C1,C2,C3,C4
      WRITE(6,1000)HX,HY,HZ
      WRITE(6,1000)HMX,HMY,HMZ
 1001         FORMAT(1X,5I5)
 1000         FORMAT(10F10.4)
      LX=NY*NZ
      LX1=2*LX
      LY=NZ*NX
      LY1=2*LY
      LZ=NX*NY
      LZ1=2*LZ
C
C
      CALL INPUT(LY,ID,NFIX,NFREE)
C     Y-DIRECTION SOLUTION
C
      CALL AKY(YK,LY,NZ,NX,HZ,HX,C1,C3)
C
C
      CALL THETA(YK,LY,HMY,YK1)
      CALL MINV(YK1,LY,D,L,M)
C
C     YK1 CONTAINS THETA INVERSE FOR KY MATRIX
C
      WRITE(6,5000)D
```

```
      CALL FORML(YK,YK1,LY,LY1,HMY,YLL)
         WRITE(6,4001)
      IWRK=LY1
      CALL TRAP(YLL,YLN,LY1,LY1)
     K=(NY-1)*NMY - 1
     DO 501 I=1,K
      CALL MULT1(YLN,YLL,LY1,TEMP)
      WRITE(6,1001)I
 501 CONTINUE
      MN=LY1*LY1
      CALL TRANS(YLN,LY1,LY1,MN,L,IWRK,IOK)
      WRITE(6,1001)IOK
 4001     FORMAT(1X,'**** THE LY MATRIX IS ****'/)
     WRITE(6,5001)
 5001 FORMAT(1X,'THE DNY MATRIX '/)
C       CALCULATION OF BK AND B23
C
     J1=0
     DO  534 J=1,LY
     IF(ID(J).EQ.0) GO TO 534
     J1=J1 + 1
     DO 535 I=1,LY
     II= I + LY
     BK(I,J1)=YLN(II,J)
535  CONTINUE
534   CONTINUE
     J1=NFREE + 1
     J2=1
     DO 536 J=1,LY
     JJ=J + LY
     IF(ID(J).EQ.1) GO TO 536
     DO 537 I=1,LY
     II=I + LY
     BK(I,J1)=YLN(II,JJ)
```

```
537     CONTINUE
       J1=J1 + 1
536     CONTINUE
C
C          FORMULATION OF B23
C
        J1=1
        DO 538 J=1,LY
        JJ=J + LY
        IF(ID(J).EQ.0) GO TO 538
         DO 539 I=1,LY
        II=I + LY
        B23(I,J1)=YLN(II,JJ)
539     CONTINUE
        J1=J1 + 1
538     CONTINUE
           CALL MINV1(BK,LY,L,M)
C
        IWRK=LY1
        MN=LY1*LY1
        CALL TRANS(YLL,LY1,LY1,MN,L,IWRK,IOK)
        WRITE(6,1001)IOK
       REWIND 2
       WRITE(2)((YK1(I,J),I=1,LY),J=1,LY),((YLL(I,J),I=1,LY1),J=1,LY1),
     * ((B23(I,J),I=1,LY),J=1,NFREE),((BK(I,J),I=1,LY),J=1,LY)
5000    FORMAT(1X,'THE DETERMINANT IS',3X,E12.4)
       STOP
       END
```

```
C...............................................................
C.                                                             .
C.                         PROGRAM MAIN3                       .
C.                PROGRAM TO FORM ZK1,ZLOC,ZLL,ZLN             .
C.                                                             .
C...............................................................
C
        IMPLICIT REAL*8(A-H,O-Z)
C
      DIMENSION ZK(140,140),ZK1(140,140),ZLOC(140,140),ZLL(280,280)
      DIMENSION ZLN(280,280),L(280),M(140),TEMP(280)
C
C
C       INPUT TO THE PROGRAM
C
        READ(5,200) NITER,NEXEC
        WRITE(6,1003) NITER
 1003   FORMAT(5X,'***** NO. OF ITERATION ALLOWED *****',I5//)
       READ(5,100)SIGMA,EPS
       READ(5,100)ALEN,WID,THIC
 100   FORMAT(3F10.0)
       READ(5,200)NX,NY,NZ,NXC
       READ(5,200)NMX,NMY,NMZ
 200   FORMAT(4I5)
C
C      CALCULATION OF VARIOUS CONSTANTS
C
       PR=1./3.
       C3=1-PR
       C1=0.5*(1-2.*PR)/C3
       C2=-0.5/C3
       C3=-PR/C3
       C4=(1+PR)*(1-2.*PR)/(1-PR)
       HX=WID/(NX-1)
```

```
      HY=ALEN/(NY-1)
      HZ=THIC/(NZ-1)
      HMX=HX/NMX
      HMY=HY/NMY
      HMZ=HZ/NMZ
       WRITE(6,1001)NX,NY,NZ,NXC
      WRITE(6,1001)NMX,NMY,NMZ
      WRITE(6,1000)ALEN,WID,THIC,EPS
      WRITE(6,1000)C1,C2,C3,C4
      WRITE(6,1000)HX,HY,HZ
      WRITE(6,1000)HMX,HMY,HMZ
 1001          FORMAT(1X,5I5)
 1000          FORMAT(10F10.4)
      LX=NY*NZ
      LX1=2*LX
      LY=NZ*NX
      LY1=2*LY
      LZ=NX*NY
      LZ1=2*LZ
C
C       Z-DIRECTION SOLUTIOON
C
      CALL AKZ(ZK,LZ,NX,NY,NXC,HX,HY,C1,C3)
      CALL THETA(ZK,LZ,HMZ,ZK1)
      CALL MINV(ZK1,LZ,D,L,M)
C
C      ZK1 CONTAINS THETA NINVERSE FOR KZ MATRIX
C
      WRITE(6,5000)D
      CALL FORML(ZK,ZK1,LZ,LZ1,HMZ,ZLL)
      CALL TRAP(ZLL,ZLN,LZ1,LZ1)
      K=(NZ - 1)*NMZ - 1
      DO 601 I=1,K
      CALL MULT1(ZLN,ZLL,LZ1,TEMP)
```

```
      WRITE(6,1001) I
 601  CONTINUE
      IWRK=LZ1
      MN=LZ1*LZ1
      CALL TRANS(ZLN,LZ1,LZ1,MN,L,IWRK,IOK)
      WRITE(6,1001)IOK
      WRITE(6,5002)
5002  FORMAT(1X,' THE DNZ MATRIX'/)
C
      DO 640 I=1,LZ
      II=LZ + I
      DO 640 J=1,LZ
      JJ=LZ + J
      ZLOC(I,J)=ZLN(II,JJ)
640   CONTINUE
      CALL MINV1(ZLOC,LZ,L,M)
C
      IWRK=LZ1
      MN=LZ1*LZ1
      CALL TRANS(ZLL,LZ1,LZ1,MN,L,IWRK,IOK)
      WRITE(6,1001)IOK
      REWIND 3
      WRITE(3)((ZK1(I,J),I=1,LZ),J=1,LZ),((ZLL(I,J),I=1,LZ1),J=1,LZ1),
     *((ZLOC(I,J),I=1,LZ),J=1,LZ)
5000  FORMAT(1X,'THE DETERMINANT IS',3X,E12.4)
      STOP
      END
```

```
C...............................................................
C.                                                             .
C.                      PROGRAM MAIN4                          .
C.               PROGRAM TO PERFORM ELASTIC                    .
C.          STRESS ANALYSIS OF A CRACKED SPECIMEN              .
C.                USING THE METHOD OF LINES                    .
C.                                                             .
C...............................................................
C
        IMPLICIT REAL*8(A-H,O-W),REAL*16(X),REAL*8(Y,Z)
C
      DIMENSION U(70,20),V(200,7),W(280,5),XK1(35,35)
      DIMENSION XLOC(35,35),XLOC1(35,35)
      DIMENSION XLL(70,70),YK1(100,100),BK(100,100)
      DIMENSION ZK1(140,140),ZLOC(140,140)
      DIMENSION ZLL(280,280),V2K(280),VNK(280),L(280),M(140),SYMBOL(2)
      DIMENSION B(280,77),TEMP(5),FN(280),B23(100,50),AK(140,20)
      DIMENSION SIGX(35,20),SIGY(100,7),SIGZ(140,5),SIGXY(35,20)
      DIMENSION SIGYZ(100,7),  SIGZX(140,5)
      DIMENSION TT(7,5,20)
      DIMENSION EPSX(35,20),EPSY(100,7),EPSZ(140,5)
      DIMENSION EPSXY(35,20),EPSYZ(100,7),EPSZX(140,5)
      DIMENSION DEPX(35,20),DEPY(100,7),DEPZ(140,5)
      DIMENSION DEPXY(35,20),DEPYZ(100,7),DEPZX(140,5)
C
       COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
       COMMON/BLK2/HX,HY,HZ,HMX,HMY,HMZ,C1,C2,C3,C4
       COMMON/BLK3/NFIX,NFREE,ID(100)
        COMMON/BLK4/IDIM1,IDIM2
      DATA SYMBOL(1)/'TENS'/,SYMBOL(2)/'COMP'/
C
C      INPUT TO THE PROGRAM
C
      READ(5,201) SYM
```

```
201    FORMAT(A4)
        READ(5,200) NITER,NEXEC
        WRITE(6,1003) NITER
 1003   FORMAT(5X,'***** NO. OF ITERATION ALLOWED *****',I5//)
       READ(5,100)SIGMA,EPS
       READ(5,100)ALEN,WID,THIC
 100   FORMAT(3F10.0)
       READ(5,200)NX,NY,NZ,NXC
       READ(5,200)NMX,NMY,NMZ
 200   FORMAT(4I5)
C
C      CALCULATION OF VARIOUS CONSTANTS
C
       PR=1./3.
       C3=1-PR
       C1=0.5*(1-2.*PR)/C3
       C2=-0.5/C3
       C3=-PR/C3
       C4=(1+PR)*(1-2.*PR)/(1-PR)
       HX=WID/(NX-1)
       HY=ALEN/(NY-1)
       HZ=THIC/(NZ-1)
       HMX=HX/NMX
       HMY=HY/NMY
       HMZ=HZ/NMZ
       WRITE(6,1001)NX,NY,NZ,NXC
       WRITE(6,1001)NMX,NMY,NMZ
       WRITE(6,1000)ALEN,WID,THIC,EPS
       WRITE(6,1000)C1,C2,C3,C4
       WRITE(6,1000)HX,HY,HZ
       WRITE(6,1000)HMX,HMY,HMZ
 1001          FORMAT(1X,5I5)
 1000          FORMAT(10F10.4)
       LX=NY*NZ
```

```
      LX1=2*LX
      LY=NZ*NX
      LY1=2*LY
      LZ=NX*NY
     NG1=NZ*LZ
     NG2=2*NG1
      LZ1=2*LZ
C
C     FIXING THE DIMENSION
C
       IF(LY.GT.LZ)GO TO 14
       IDIM1=LZ
       IDIM2=LZ1
       GO TO 12
 14    IDIM1=LY
       IDIM2=LY1
C
C
C       FORMATION OF ID ARRAY
C
12      CALL INPUT(LY,ID,NFIX,NFREE)
C
       IF(NEXEC.EQ.0) GO TO 11
C
C
C     FORMULATION OG SOLUTION ALGORITHM
C
C     INITIALIZATION OF U,V,W MATRICES
      CALL INIT(U,LX1,NX)
      CALL INIT(V,LY1,NY)
      CALL INIT(W,LZ1,NZ)
C
C
      REWIND 2
```

```
      READ(2)((YK1(I,J),I=1,LY),J=1,LY),((ZLL(I,J),I=1,LY1),J=1,LY1),
     * ((B23(I,J),I=1,LY),J=1,NFREE),((BK(I,J),I=1,LY),J=1,LY)
      REWIND 3
      READ(3)((ZK1(I,J),I=1,LZ),J=1,LZ),((ZLL(I,J),I=1,LZ1),J=1,LZ1),
     * ((ZLOC(I,J),I=1,LZ),J=1,LZ)
      REWIND 1
      READ(1)((XK1(I,J),I=1,LX),J=1,LX),((XLL(I,J),I=1,LX1),J=1,LX1),
     * ((XLOC1(I,J),I=1,LX),J=1,LX),((XLOC(I,J),I=1,LX),J=1,LX)
C
C
C         START JF ITERATION
C
       ITER=1
C
C                INITIALIZE TEMP ARRAY
C
                 DO 60 I=1,NZ
 60              TEMP(I)=0.0
 700      WRITE(6,5008) ITER
 5008     FORMAT(5X,'***** THE ITERATION NO. ******',I5//)
C
C         Y-DIR SOLUTION
C
      REWIND 2
      READ(2)((YK1(I,J),I=1,LY),J=1,LY),((ZLL(I,J),I=1,LY1),J=1,LY1)
C
      IF(SYM.EQ.SYMBOL(1)) GO TO 1
      GO TO 2
1     CALL CVDISP(U,V,W,AK,YK1,ZLL,B23,V2K,VNK,BK,L,M,B,FN)
      GO TO 3
2         CALL CCV(U,V,W,AK,YK1,ZLL,B23,V2K,VNK,BK,L,M,B,FN)
C
C           WRITE VDISPLACEMENT AT Y=0 PLANE
C
```

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```
C        WRITE(6,1000)(V(J,1),J=1,LY)
C
C           WRITE ANSWER FOR V DISP
C
C
C           Z- DIR SOLUTION
C
3        REWIND 3
         READ(3)((ZK1(I,J),I=1,LZ),J=1,LZ),((ZLL(I,J),I=1,LZ1),J=1,LZ1)
            CALL WDISP(U,V,W,AK,ZK1,ZLL,V2K,VNK,L,M,B,ZLOC,FN)
C
C           OUTPUT FOR W DISPLACEMENT
C
C
C           SOLUTION FOR U DISP
C
         CALL UDISP(U,V,W,AK,XK1,XLL,XLOC1,V2K,VNK,L,M,B,XLOC,FN)
C
C             CHECK FOR CONVERGENCE
C
              BIG=0.0
              DO 61 I=1,NZ
              A=V(I,1) - TEMP(I)
              IF(DABS(A).GT.BIG)BIG=A
 61           CONTINUE
           WRITE(6,1000)BIG
C
              IF(BIG.LT.EPS)GO TO 65
C
              DO 62 I=1,NZ
              TEMP(I)=V(I,1)
 62           CONTINUE
C
C        WRITE ANSWER FOR U
```

```
C
        ITER=ITER + 1
          WRITE(6,1000)(V(1,J),J=1,NY)
        IF(ITER.LE.NITER)GO TO 700
C
                 ITER=ITER -1
 65              WRITE(6,5008) ITER
CWRITE DISPLACEMENTS ON UNIT =2
C
C         WRITE THE SLOPES
C
C
C
C
          IHEAD=2
           CALL OUTPUT(V,LY1,NY,IHEAD,1,LY)
      IHEAD=3
      CALL OUTPUT(W,LZ1,NZ,IHEAD,1,LZ)
      IHEAD=1
      CALL OUTPUT(U,LX1,NX,IHEAD,1,LX)
C
C             CALCULATION OF SIGMA Y
C
C
C
        CALL STRESS(U,V,W,LX,LY,LZ,NX,NY,NZ,PR,SIGX,LX1,LY1,LZ1)
      CALL SHEAXY(U,V,LX1,LY1,HY,HX,PR,LX,SIGXY,NX,NY,NZ,1)
           CALL STRESS(V,W,U,LY,LZ,LX,NY,NZ,NX,PR,SIGY,LY1,LZ1,LX1)
      CALL SHEAYZ(V,W,LY1,LZ1,HZ,HY,PR,LY,SIGYZ,NY,NZ,NX,1)
       CALL STRESS(W,U,V,LZ,LX,LY,NZ,NX,NY,PR,SIGZ,LZ1,LX1,LY1)
      CALL SHEAYZ(W,U,LZ1,LX1,HX,HZ,PR,LZ,SIGZX,NZ,NX,NY,1)
C
      DO 71 I=1,LX
71    WRITE(6,5009)(SIGX(I,J),J=1,NX)
```

```
      DO 72 I=1,LY
72    WRITE(6,5009)(SIGY(I,J),J=1,NY)
      DO 73 I=1,LZ
73    WRITE(6,5009)(SIGZ(I,J),J=1,NZ)
      DO 74 I=1,LX
74    WRITE(6,5009)(SIGXY(I,J),J=1,NX)
      DO 75 I=1,LY
75    WRITE(6,5009)(SIGYZ(I,J),J=1,NY)
      DO 76 I=1,LZ
76    WRITE(6,5009)(SIGZX(I,J),J=1,NZ)
C
 5009          FORMAT(5X,10F12.6)
C
C
 5000  FORMAT(1X,'THE DETERMINANT IS',3X,E12.4)
C
C     CALCULATION OF EFFECTIVE STRESS
C
      CALL EFTRES(SIGX,SIGY,SIGZ,SIGXY,SIGYZ,SIGZX,TT,1)
      WRITE(6,5013)
5013  FORMAT(5X,'THE EFFECTIVE STRESS MATRIX')
      WRITE(6,5009)(((TT(I,J,K),I=1,NY),J=1,NZ),K=1,NX)
C
C      FIND MAX EFFECTIVE STRESS
C
      AMAX=TT(1,1,1)
      DO 95 I=1,NX
      DO 95 J=1,NZ
      DO 95 K=1,NY
95    IF(TT(K,J,I).GT.AMAX)AMAX=TT(K,J,I)
C
      WRITE(6,5009)AMAX
C
C
```

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```
C      SCALING OF THE STRESSES AND DISPLACEMENT
C
      FAC=1./AMAX
      CALL SCAVE1(U,FAC,NG2)
      CALL SCAVE1(V,FAC,NG2)
      CALL SCAVE1(W,FAC,NG2)
C
C     SET TT=1.0
C
      DO 96 I=1,NX
      DO 96 J=1,NZ
      DO 96 K=1,NY
96    TT(K,J,I)=1.0
      CALL MATIN(EPSX,NG1)
      CALL MATIN(EPSY,NG1)
      CALL MATIN(EPSZ,NG1)
      CALL MATIN(EPSXY,NG1)
      CALL MATIN(EPSYZ,NG1)
      CALL MATIN(EPSZX,NG1)
      CALL MATIN(DEPX,NG1)
      CALL MATIN(DEPY,NG1)
      CALL MATIN(DEPZ,NG1)
      CALL MATIN(DEPXY,NG1)
      CALL MATIN(DEPYZ,NG1)
      CALL MATIN(DEPZX,NG1)
      IINCR=0
      REWIND 4
      WRITE(4)U,V,W,EPSX,EPSY,EPSZ,EPSXY,EPSYZ,EPSZX,
     1TT,IINCR,FAC,DEPX,DEPY,DEPZ,DEPXY,DEPYZ,DEPZX
C
11    STOP
      END
```

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```
C       INPUT TO THE PROGRAM
C
      READ(5,201) SYM
201   FORMAT(A4)
        READ(5,200) NITER,NEXEC
        WRITE(6,1003) NITER
1003  FORMAT(5X,'***** NO. OF ITERATION ALLOWED *****',I5)
       READ(5,100)EPS,EPS1
      WRITE(6,300)EPS,EPS1
300   FORMAT(5X,'EPS=',F10.5,'EPS1=',F10.5)
       READ(5,100)ALEN,WID,THIC
 100   FORMAT(3F10.0)
       READ(5,200)NX,NY,NZ,NXC
       READ(5,200)NMX,NMY,NMZ
 200   FORMAT(4I5)
C
C      CALCULATION OF VARIOUS CONSTANTS
C
       PR=1./3.
       C3=1-PR
       C1=0.5*(1-2.*PR)/C3
       C2=-0.5/C3
       C3=-PR/C3
       C41=(1+PR)*(1-2.*PR)/(1-PR)
       HX=WID/(NX-1)
       HY=ALEN/(NY-1)
       HZ=THIC/(NZ-1)
       HMX=HX/NMX
       HMY=HY/NMY
       HMZ=HZ/NMZ
       WRITE(6,1001)NX,NY,NZ,NXC
       WRITE(6,1001)NMX,NMY,NMZ
       WRITE(6,1000)ALEN,WID,THIC,EPS
       WRITE(6,1000)C1,C2,C3,C41
```

```
      WRITE(6,1000)HX,HY,HZ
      WRITE(6,1000)HMX,HMY,HMZ
 1001          FORMAT(1X,5I5)
 1000          FORMAT(10F10.4)
      LX=NY*NZ
      LX1=2*LX
      LY=NZ*NX
      LY1=2*LY
      LZ=NX*NY
      LZ1=2*LZ
     NG1=NZ*LZ
     NG2=2*NG1
C
C     FIXING THE DIMENSION
C
       IF(LY.GT.LZ)GO TO 14
       IDIM1=LZ
       IDIM2=LZ1
       GO TO 12
 14    IDIM1=LY
       IDIM2=LY1
C
C
C       FORMATION OF ID ARRAY
C
12      CALL INPUT(LY,ID,NFIX,NFREE)
C
       IF(NEXEC.EQ.0) GO TO 11
C
C
C     FORMULATION OG SOLUTION ALGORITHM
C
C
C
```

```
      REWIND 2
      READ(2)((YK1(I,J),I=1,LY),J=1,LY),((ZLL(I,J),I=1,LY1),J=1,LY1),
     * ((B23(I,J),I=1,LY),J=1,NFREE),((BK(I,J),I=1,LY),J=1,LY)
      REWIND 3
      READ(3)((ZK1(I,J),I=1,LZ),J=1,LZ),((ZLL(I,J),I=1,LZ1),J=1,LZ1),
     * ((ZLOC(I,J),I=1,LZ),J=1,LZ)
      REWIND 1
      READ(1)((XK1(I,J),I=1,LX),J=1,LX),((XLL(I,J),I=1,LX1),J=1,LX1),
     * ((XLOC1(I,J),I=1,LX),J=1,LX),((XLOC(I,J),I=1,LX),J=1,LX)
C
C
C      RAED ALL THE MATRICES
C
      REWIND 4
      READ(4)U,V,W,EPSX,EPSY,EPSZ,EPSXY,EPSYZ,EPSZX,
     *TT,IINCR,R3
C
      INCR=1
      READ(5,5010)NINCR,AINCR
      WRITE(6,5011)NINCR,AINCR
5011  FORMAT(5X,'NINCR=',I5,'AINCR=',F10.5)
5010  FORMAT(I5,F10.0)
C
C
C          START OF ITERAT'ON
C
701        ITER=1
C
C                 INITIALIZE TEMP ARRAY
C
      CALL MATIN(TEMP1,NZ)
      CALL MATIN(TEMP,NG1)
5008   FORMAT(5X,'**** THE ITERATION NO. ****',I5)
C
```

```
      CALL MATIN(DEFX,NG1)
      CALL MATIN(DEFY,NG1)
      CALL MATIN(DEFZ,NG1)
      CALL MATIN(DEFXY,NG1)
      CALL MATIN(DEFYZ,NG1)
      CALL MATIN(DEFZX,NG1)
      R4=R3 + AINCR
C
C
C         Y-DIR SOLUTION
C
700      REWIND 2
       READ(2)((YK1(I,J),I=1,LY),J=1,LY),((ZLL(I,J),I=1,LY1),J=1,LY1)
C
      IF(SYM.EQ.SYMBOL(1)) GO TO 1
      GO TO 2
1     CALL PVDISP(U,V,W,AK,YK1,ZLL,B23,V2K,VNK,BK,L,M,B,FN
     1,EPSY,DEPY)
      GO TO 3
2         CALL PCCV(U,V,W,AK,YK1,ZLL,B23,V2K,VNK,BK,L,M,B,FN
     1,EPSY,DEPY,R4,EPSYZ,DEPYZ,EPSXY,DEPXY,DR,DR1,DR2)
C
C          Z- DIR SOLUTION
C
3      REWIND 3
       READ(3)((ZK1(I,J),I=1,LZ),J=1,LZ),((ZLL(I,J),I=1,LZ1),J=1,LZ1)
          CALL PWDISP(U,V,W,AK,ZK1,ZLL,V2K,VNK,L,M,B,ZLOC,FN
     1,EPSZ,DEPZ,EPSZX,DEPZX,EPSYZ,DEPYZ,DR,DR1,DR2)
C
      WRITE(6,5008) ITER
C           OUTPUT FOR W DISPLACEMENT
C
C           SOLUTION FOR U DISP
C
```

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```
      CALL FUDISP(U,V,W,AK,XK1,XLL,XLOC1,V2K,VNK,L.M,B,XLOC,FN
     1,EPSX,DEPX,EPSXY,DEPXY,EPSZX,DEPZX,DR,DR1,DR2)
C
C      CALCULATION OF TOTAL SHEARING STRAINS
C
      CALL SHEAXY(U,V,LX1,LY1,HY,HX,PR,LX,EPXY,NX,NY,NZ,0)
      CALL SHEAYZ(V,W,LY1,LZ1,HZ,HY,PR,LY,EPYZ,NY,NZ,NX,0)
      CALL SHEAYZ(W,U,LZ1,LX1,HX,HZ,PR,LZ,EPZX,NZ,NX,NY,0)
C
      DO 81 I=1,NX
      DO 81 J=1,LX
      II=J + LX
      EPX(J,I)=U(II,I) - EPSX(J,I)
      EPXY(J,I)=EPXY(J,I) - EPSXY(J,I)
81    CONTINUE
C
      DO 82 I=1,NY
      DO 82 J=1,LY
      II=J + LY
      EPY(J,I)=V(II,I) - EPSY(J,I)
      EPYZ(J,I)=EPYZ(J,I) - EPSYZ(J,I)
82    CONTINUE
C
      DO 83 I=1,NZ
      DO 83 J=1,LZ
      II=J + LZ
      EPZ(J,I)=W(II,I) - EPSZ(J,I)
      EPZX(J,I)=EPZX(J,I) - EPSZX(J,I)
83    CONTINUE
C
C      CALCULATION OF EFFECTIVE STRAIN MATRIX
C
      CALL EFTRES(EPX,EPY,EPZ,EPXY,EPYZ,EPZX,EET,0)
C
```

```
C       CALCULATION OF DELEP
C
      DO 84 I=1,NX
      DO 84 J=1,NZ
      DO 84 K=1,NY
      A12=EET(K,J,I) -2.0*(1.+PR)*TT(K,J,I)/3.0
      IF(A12.LE.0.0) GO TO 60
      CALL EH(TT(K,J,I),AA)
      A11=1.0/(1.+2.*(1.+PR)*AA/3.0)
      DELEP(K,J,I)=A11*A12
      GO TO 84
60    DELEP(K,J,I)=0.D0
84     CONTINUE
C
C     CALCULATION OF PLASTIC STRAINS
C
      CALL PLSTRN(EET,DELEP,EPX,EPY,EPZ,EPXY,EPYZ,EPZX,DEPX,DEPY,
     1DEPZ,DEPXY,DEPYZ,DEPZX)
C
C      CHECK FOR CONVERGENCE
C
      BIG=0.0
      DO 88 I=1,NZ
      A=DABS(V(I,1) - TEMP1(I))
      IF(A.GT.BIG)BIG=A
88    CONTINUE
      WRITE(6,1005)BIG
1005  FORMAT(F10.7)
C
      NN=0
      SUM1=0.0
      SUM=0.0
      DO 85 I=1,NY
      DO 85 J=1,LY
```

```
      IF(DEPY(J,I).EQ.0.0) GO TO 85
      NN=NN + 1
      SUM1=SUM1 + DABS(DEPY(J,I) - TEMP(J,I))
      SUM=SUM + DABS((DEPY(J,I) - TEMP(J,I))/DEPY(J,I))
85    CONTINUE
      SUM=SUM/(NN)
      SUM1=SUM1/NN
      WRITE(6,5009)(DEPY(J,1),J=50,55)
      WRITE(6,5009)(TEMP(J,1),J=51,55)
      WRITE(6,1004)SUM,SUM1,(V(1,I),I=1,NY)
1004  FORMAT(1X,2F10.5,'VDISP2',7F10.5)
      IF(SUM.LE.EPS1.AND.BIG.LE.EPS)GO TO 86
        ITER=ITER + 1
C
      DO 90 I=1,NZ
      TEMP1(I)=V(I,1)
90    CONTINUE
      CALL MATEQU(DEPY,NG1,TEMP)
C
        IF(ITER.LE.NITER)GO TO 700
C
                ITER=ITER -1
86    IINCR=IINCR+1
      WRITE(6,5012)IINCR
5012  FORMAT(//5X,'COMPLETED INCREMENT=',I5)
C
      INCR=INCR + 1
C
C     UPDATWE THE PLASTIC STRAINS
C
      CALL MATADD(DEPX,NG1,EPSX)
      CALL MATADD(DEPY,NG1,EPSY)
      CALL MATADD(DEPZ,NG1,EPSZ)
      CALL MATADD(DEPXY,NG1,EPSXY)
```

```
      CALL MATADD(DEPYZ,NG1,EPSYZ)
      CALL MATADD(DEPZX,NG1,EPSZX)
C
        CALL STRESS(U,V,W,LX,LY,LZ,NX,NY,NZ,PR,SIGX,LX1,LY1,LZ1)
      CALL SHEAXY(U,V,LX1,LY1,HY,HX,PR,LX,SIGXY,NX,NY,NZ,1)
C
      CC1=1./(1. + PR)
C
      DO 94 I=1,NX
      DO 94 J=1,LX
      SIGX(J,I)=SIGX(J,I) - CC1*EPSX(J,I)
      SIGXY(J,I)=SIGXY(J,I) - CC1*EPSXY(J,I)
94    CONTINUE
            CALL STRESS(V,W,U,LY,LZ,LX,NY,NZ,NX,PR,SIGY,LY1,LZ1,LX1)
      CALL SHEAYZ(V,W,LY1,LZ1,HZ,HY,PR,LY,SIGYZ,NY,NZ,NX,1)
C
      DO 91 I=1,NY
      DO 91 J=1,LY
      SIGY(J,I)=SIGY(J,I) - CC1*EPSY(J,I)
      SIGYZ(J,I)=SIGYZ(J,I) - CC1*EPSYZ(J,I)
91    CONTINUE
       CALL STRESS(W,U,V,LZ,LX,LY,NZ,NX,NY,PR,SIGZ,LZ1,LX1,LY1)
      CALL SHEAYZ(W,U,LZ1,LX1,HX,HZ,PR,LZ,SIGZX,NZ,NX,NY,1)
      DO 92 I=1,NZ
      DO 92 J=1,LZ
      SIGZ(J,I)=SIGZ(J,I) - CC1*EPSZ(J,I)
      SIGZX(J,I)=SIGZX(J,I) - CC1*EPSZX(J,I)
92    CONTINUE
C
C      UPDATING OF EFFECTIVE STRESS MATRIX
C
      CALL EFTRES(SIGX,SIGY,SIGZ,SIGXY,SIGYZ,SIGZX,TT,1)
      WRITE(6,5013)
5013  FORMAT(5X,' THE EFFECTIVE STRESSS MATRIX')
```

```
C
C       FIND MAX EFFECTIVE STRESS
C
      AMAX=TT(1,1,1)
      DO 95 I=1,NX
      DO 95 J=1,NZ
      DO 95 K=1,NY
95     IF(TT(K,J,I).GT.AMAX)AMAX=TT(K,J,I)
C
C       ADJUSTMENT OF TT FOR UNYIELDED NODES
C
      DO 93 I=1,NX
      DO 93 J=1,NZ
      DO 93 K=1,NY
      IF(DELEP(K,J,I).GT.0.D0) GO TO 93
      TT(K,J,I)=1.0
93    CONTINUE
C
      WRITE(6,5009)(V(1,J),J=1,NY)
      WRITE(6,5009)(SIGY(I,1),I=1,LY)
      WRITE(6,5009)(SIGY(I,NY),I=1,LY)
C
      R5=3.7251*R4
      WRITE(6,5009)R4,R5,AMAX
      DO 72 I=1,NX
      DO 72 J=1,NZ
      DO 72 K=1,NY
      IF(DELEP(K,J,I).LE.0.D0)GO TO 72
      K1=(J -1)*NY + K
      K2=(I - 1)*NZ + J
      K3=(K -1)*NX + I
      WRITE(6,5015)K,J,I,DELEP(K,J,I),TT(K,J,I),EPSX(K1,I),EPSY(K2,K)
     1,EPSZ(K3,J),EPSXY(K1,I),EPSYZ(K2,K),EPSZX(K3,J)
72    CONTINUE
```

```
5015  FORMAT(3I5,8F12.5)
C
      R3=R4
      REWIND 7
      WRITE(7)U,V,W,EPSX,EPSY,EPSZ,EPSXY,EPSYZ,EPSZX,
     1TT,IINCR,R4
      IF(INCR.LE.NINCR) GO TO 701
      STOP
 5000  FORMAT(1X,'THE DETERMINANT IS',3X,E12.4)
 5009          FORMAT(5X,10F12.6)
11      STOP
       END
```

```
      SUBROUTINE UDISP(U,V,W,AK,XK1,XLL,XLOC1,V2K,VNK,L,M,B,XLOC,FN)
       IMPLICIT REAL*8(A-H,O-W),REAL*16(X),REAL*8(Y,Z)
      COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
      COMMON/BLK2/HX,HY,HZ,HMX,HMY,HMZ,C1,C2,C3,C4
       COMMON/BLK4/IDIM1,IDIM2
         DIMENSION U(LX1,1),V(LY1,1),W(LZ1,1),AK(IDIM1,1),XLL(LX1,1)
         DIMENSION XLOC1(LX,1),VNK(1),L(1),M(1),B(IDIM2,1),V2K(1)
       DIMENSION XK1(LX,1)
      DIMENSION XV2K(98),XVNK(98)
C
C        IN THIS SUBROUTINE XLN HAS BEEN TAKEN OUT AND IS REPLACED BY
C        XLOC1(LX,1) WHICH IS ALPHA4
C
         DIMENSION XLOC(LX,1),FN(1)
C     GENERATION OF R VECTORS
C
C
       CALL FORMR(V,W,NX,LX,NY,LY,LZ,LY1,LZ1,C2,AK,IDIM1)
        C5=C1/(6.*HY)
        C6=11.*C5
       C7=C1/(6.*HZ)
        C8=11.*C7
       N10=NY - 1
       JJ=0
        II=1
        DO 71 I=1,NX
         DO 72 J=1,LX
         JJ=JJ + 1
          IF(JJ.EQ.1)AK(J,I)=AK(J,I) - C6*V(II,JJ)
          IF(JJ.EQ.2)AK(J,I)=AK(J,I) + C5*V(II,JJ-1)
          IF(JJ.EQ.N10)AK(J,I)=AK(J,I) - C5*V(II,JJ+1)
          IF(JJ.NE.NY)GO TO 72
         AK(J,I)=AK(J,I) + C6*V(II,JJ)
         JJ=0
```

```
            II=II + 1
 72         CONTINUE
 71         CONTINUE
          N11=(NZ -2)*NY + 1
          N12=(NZ-1)*NY
          N14=NZ-1
          DO 73 I=1,NX
          MM=I
           DO 74 J=N11,N12
            AK(J,I)=AK(J,I) - C7*W(MM,NZ)
            N13=J + NY
            AK(N13,I)=AK(N13,I) + C8*W(MM,NZ)
              MM= MM + NX
 74          CONTINUE
  73         CONTINUE
C
        CALL DORMRF(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2)
C
C      GENERATION OF INITIAL CONDITION ON FACE FOUR
C
       II=LY + 1
       JJ=0
        MM=LZ + 1
        KK=1
       DO 280 J=1,LX
       JJ=JJ + 1
        XV2K(J)=C3*(V(II,JJ) + W(MM,KK))
        MM=MM + NX
       IF(JJ.NE.NY) GO TO 280
       JJ=0
        KK=KK + 1
        MM=LZ +1
       II=II + 1
 280   CONTINUE
```

```
C
C       GENERATION OF INITIAL CONDITION ON FACE ONE
C
        II=LY + (NX-1)*NZ + 1
        JJ=0
         MM=LZ + NX
         KK=1
        DO 310 J=1,LX
        JJ=JJ + 1
         XVNK(J)=C3*(V(II,JJ) + W(MM,KK))
         MM=MM + NX
        IF(JJ.NE.NY) GO TO 310
        JJ=0
         KK=KK + 1
         MM=LZ + NX
        II=II + 1
 310    CONTINUE
C
CSET LAST ELEMENT OF VNK AS ZERO
C
           TEMP=XVNK(LX)
           XVNK(LX)=0.0
C
         K3=(NX - 1)*NMX + 1
C
C
               B(LX1,K3)=B(LX,K3)
C
C
C       CALCULATION FOR UNKNOWN PART OF VECTOR U
C
C       INITIALIZATION OF PART OF B TO BE USED AS WORKSPACE
C
         K1=(NX - 1)*NMX + 1
```

```
      DO 360 I=1,LX
      II=I + LX
      U(II,1)=XV2K(I)
      XTEM=0.0
      DO 361 K=1,LX
      XTEM=XTEM + XLOC1(I,K)*XV2K(K)
 361  CONTINUE
      XVNK(I)=XVNK(I) - XTEM - B(II,K1)
 360  CONTINUE
C
      DO 380 I=1,LX
      XTEM=0.0
      DO 381 K=1,LX
      XTEM=XTEM + XLOC(I,K)*XVNK(K)
 381  CONTINUE
      U(I,1)=XTEM
 380  CONTINUE
C
      CALL DFORMD(XLL,B,NX,NMX,LX1,U,XV2K,XVNK,IDIM2)
C
       U(LX1,NX)=TEMP
 5000 FORMAT(1X,'THE VALUE OF DETERMINANT IS',3X,E12.4)
 6000 FORMAT(1X,'THIS STEP IS DONE',3X,I4)
      RETURN
      END
```

```
      SUBROUTINE CCV(U,V,W,AK,YK1,XLL,B23,V2K,VNK,BK,L,M,B,FN)
       IMPLICIT REAL*8(A-H,O-Z)
      COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
      COMMON/BLK2/HX,HY,HZ,HMX,HMY,HMZ,C1,C2,C3,C4
       COMMON/BLK3/NFIX,NFREE,ID(1)
       COMMON/BLK4/IDIM1,IDIM2
         DIMENSION U(LX1,1),V(LY1,1),W(LZ1,1),AK(IDIM1,1),XLL(IDIM2,1)
C     DIMEN OF XLL IS CHANGE TO ACCOM. LY NE.LZ CASE
         DIMENSION B23(LY,1),VNK(1),L(1),M(1),B(IDIM2,1),YK1(LY,1)
       DIMENSION V2K(1)
         DIMENSION BK(LY,1),FN(1)
C     GENERATION OF R VECTORS
C
C
       CALL FORMR(W,U,NY,LY,NZ,LZ,LX,LZ1,LX1,C2,AK,IDIM1)
C
C        GENERATION OF MODIFIED R
C
       C5=C1/(6.*HX)
       C6=11.*C5
        C7=C1/(6.*HZ)
        C8=11.*C7
        JJ=0
        N10=NZ - 1
        II=1
         DO 602 I=1,NY
         DO 603 J=1,LY
          JJ= JJ + 1
          IF(JJ.EQ.N10)AK(J,I)=AK(J,I) - C7*W(II,JJ+1)
          IF(JJ.NE.NZ) GO TO 603
           AK(J,I)=AK(J,I) + C8*W(II,JJ)
           JJ=0
           II= II + 1
 603     CONTINUE
```

```
602      CONTINUE
          N11=(NX - 2)*NZ
          N12=N11 + NZ
           N13=NX - 1
         DO 605 I=1,NY
          MM=I
          DO 604 J=1,NZ
           AK(J,I)=AK(J,I) - C6*U(MM,1)
           J1=NZ + J
           AK(J1,I)=AK(J1,I) + C5*U(MM,1)
           J2=N11 + J
           AK(J2,I)=AK(J2,I) - C5*U(MM,NX)
           J3=N12 + J
           AK(J3,I)=AK(J3,I) + C6*U(MM,NX)
           MM=MM + NY
604       CONTINUE
605       CONTINUE
C
       CALL RF(YK1,XLL,HY,HMY,NY,NMY,LY,LY1,B,AK,IDIM1,IDIM2,NZ,C1,HX)
C
C      GENERATION OF INITIAL CONDITIONS ON FACE FIVE
C
      II=LZ+1
      JJ=0
       MM=LX + 1
       KK=1
       J1=1
      K3=NZ*NXC
      DO 528 J=1,LY
      JJ=JJ+1
       IF(ID(J).EQ.0)GO TO 30
       V2K(J1)=C3*(W(II,JJ) + U(MM,KK))
       J1=J1 + 1
30     MM=MM + NY
```

```
      IF(JJ.NE.NZ) GO TO 528
      JJ=0
       KK=KK + 1
       MM=LX + 1
      II=II+1
 528  CONTINUE
C
C     GENERATTION OF INITIAL CONDITIONS ON FACE TWO
C
      II=LZ + (NY-1)*NX + 1
      JJ=0
       MM=LX + NY
       KK=1
      DO 531 J=1,LY
      JJ=JJ+1
       VNK(J)=  C3*(W(II,JJ) + U(MM,KK))
       MM=MM + NY
      IF(JJ.NE.NZ) GO TO 531
      JJ=0
       KK=KK + 1
       MM=LX + NY
      II=II + 1
 531  CONTINUE
C
C     CALCULATION FOR UNKNOWN PART OF VECTOR V
C
C     PART OF MATRIX B IS INITIALIZED
C
C
C     CALCULATION OF VARIOUS SUB MATRICE
C
      K1=NZ*NXC
      K2=NZ*(NX-NXC)
      K3=LY + K1
```

```
      K4=(NY-1)*NMY + 1
C
       DO 536 I=1,LY
       II=I + LY
       TEMP=0.0
       DO 550 K=1,NFREE
       TEMP=TEMP + B23(I,K)*V2K(K)
 550   CONTINUE
       VNK(I)=VNK(I) - TEMP - B(II,K4)
 536   CONTINUE
C
       DO 538 I=1,LY
       TEMP=0.0
       DO 551 K=1,LY
 551   TEMP=TEMP + BK(I,K)*VNK(K)
 538   FN(I)=TEMP
C
C
C      FORMULATION OF V1 VECTOR
C
       I2=1
       I1=1
       DO 545 I=1,LY
       IF(ID(I).EQ.0) GO TO 700
       V(I,1)=FN(I1)
       II=I + LY
       V(II,1)=V2K(I1)
       I1=I1 + 1
       GO TO 545
700    II= I + LY
       V(II,1)=FN(NFREE + I2)
       I2=I2 + 1
545    CONTINUE
C
```

```
C
      CALL FORMD(XLL,B,NY,NMY,LY1,V,V2K,VNK,IDIM2)
 5000 FORMAT(1X,'THE VALUE OF DETERMINANT IS',E12.4)
 6000  FORMAT(1X,'STEP IS DONE',3X,I4)
      RETURN
      END
```

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```
      SUBROUTINE WDISP(U,V,W,AK,ZK1,XLL,V2K,VNK,L,M,B,ZLOC,FN)
C
C         IN THIS FILE XLN=ZLN HAS BEEN TAKEN OUT FOR SAVING SP
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
      COMMON/BLK2/HX,HY,HZ,HMX,HMY,HMZ,C1,C2,C3,C4
       COMMON/BLK4/IDIM1,IDIM2
         DIMENSION U(LX1,1),V(LY1,1),W(LZ1,1),AK(IDIM1,1),XLL(ID
         DIMENSION VNK(1),L(1),M(1),B(IDIM2,1)
         DIMENSION ZK1(LZ,1),ZLOC(LZ,1),FN(1),V2K(1)
C     GENERATION OF R VECTORS
C
C
       CALL FORMR(U,V,NZ,LZ,NX,LX,LY,LX1,LY1,C2,AK,IDIM1)
C
C      ********* GENERATION OF MODIFIED R
C
         C5=C1/(6.*HX)
          C6=11.*C5
         C7=C1/(6.*HY)
          C8=11.*C7
         N10=NX -1
         II=1
          JJ=0
         DO 102 I=1,NZ
         DO 103 J=1,LZ
          JJ=JJ + 1
           IF(JJ.EQ.1)AK(J,I)=AK(J,I) - C6*U(II,JJ)
           IF(JJ.EQ.2)AK(J,I)=AK(J,I) + C5*U(II,JJ-1)
           IF(JJ.EQ.N10)AK(J,I)=AK(J,I) - C5*U(II,JJ+1)
           IF(JJ.NE.NX) GO TO 103
           AK(J,I)=AK(J,I) + C6*U(II,JJ)
           JJ=0
```

```
              II=II + 1
  103        CONTINUE
  102       CONTINUE
           DO 104 I=1,NZ
           MM=I
           DO 105 J=1,NXC
           AK(J,I)=AK(J,I) - C8*V(MM,1)
           J1=J + NX
            AK(J1,I)=AK(J1,I) + C7*V(MM,1)
             MM=MM + NZ
  105      CONTINUE
  104      CONTINUE
             N11=(NY -2)*NX
             N12=N11 + NX
             N13=NY -1
            DO 106 I=1,NZ
             MM=I
            DO 107 J=1,NX
              J2=N11 + J
               AK(J2,I)=AK(J2,I) - C7*V(MM,NY)
              J3=N12 + J
                AK(J3,I)=AK(J3,I) + C8*V(MM,NY)
               MM= MM + NZ
  107        CONTINUE
  106        CONTINUE
C
C
        CALL FORMRF(ZK1,XLL,HZ,HMZ,NZ,NMZ,LZ,LZ1,B,AK,IDIM1,IDIM2)
C
C      GENERATION OF INITIAL CONDITION OF FACE SIX
C
C
C      GENERATION OF INITIAL CONDITION ON FACE THREE
C
```

```
      II=LX + (NZ-1)*NY + 1
      JJ=0
       MM=LY + NZ
       KK=1
      DO 628 J=1,LZ
      JJ=JJ + 1
       VNK(J)=C3*(U(II,JJ) + V(MM,KK))
       MM=MM + NZ
      IF(JJ.NE.NX) GO TO 628
      JJ=0
       KK=KK + 1
       MM=LY + NZ
      II=II + 1
 628  CONTINUE
C
C     CALCULATION OF UNKNOWN PART PF VECTOR W
C
C     YK1 CONTAINS L4-1
C    INITIALIZATION OF PART OF B TO BE USED AS WORKSPACE
      K1=(NZ-1)*NMZ + 1
       DO 660 I=1,LZ
       II=LZ + I
       VNK(I)=VNK(I) - B(II,K1)
 660   CONTINUE
       DO 670 I=1,LZ
       TEMP=0.0
       II=I + LZ
       DO 671 K=1,LZ
       TEMP=TEMP + ZLOC(I,K)*VNK(K)
 671   CONTINUE
 670   W(II,1)=TEMP
C
       CALL FORMD(XLL,B,NZ,NMZ,LZ1,W,V2K,VNK,IDIM2)
 5000 FORMAT(1X,'THE VALUE OF DETERMINANT IS',3X,E12.4)
```

```
6000  FORMAT(1X,'STEP IS DONE',3X,I4)
      RETURN
      END
```

```
      SUBROUTINE DAKX(XK,LX,NY,NZ,HY,HZ,C1,C3)
C
C
C     A SUBROUTINE TO CALCULATE KX MATRIX
C
         IMPLICIT REAL*16(A-H,O-Z)
         DIMENSION XK(LX,1)
C
      A1=C3/(HY*HY)
      A2=C1*A1/C3
      A11=C3/(HZ*HZ)
      A12=C1*A11/C3
C
C
C     INIALIZE XK MATRIX
      K1=NY*NZ
       DO 10 I=1,K1
       DO 10 J=1,K1
   10   XK(I,J)=0.0
         TEMP1=A2/12.
         TEMP2=-4.*A2/3.
         XK(1,1)=5.*A2/3.
         XK(1,2)=-17.*A2/12.
         XK(1-3)=-A2/3.
         XK(1,4)=TEMP1
         XK(2,1)=TEMP2
         XK(2,2)=31.*A2/12.
         XK(2,3)=TEMP2
         XK(2,4)=TEMP1
         XK(3,1)=TEMP1
         XK(3,2)=TEMP2
         XK(3,3)=5.*A2/2.
         XK(3,4)=TEMP2
         XK(3,5)=TEMP1
```

```
      K1= NY -5
      K2=2
      K3=6
      K4=0
      IF(K1.EQ.0) GO TO 40
        DO 20 I=1,K1
        DO 30 J=K2,K3
        K4=K4 + 1
        XK(I+3,J)= XK(3,K4)
 30     CONTINUE
        K2=K2 + 1
        K3=K3 +1
        K4=0
 20     CONTINUE
C
C     FILL THE LAST TWO LINES
C
 40     DO 50 I=1,2
        DO 50 J=1,5
 50     XK(NY-2 +I,NY -5+J)=XK(3-I,6-J)
        DO 90 I=1,NY
 90     WRITE(6,77)(XK(I,J),J=1,NY)
 77   FORMAT(1X,12F10.4)
      N=NZ-1
        DO 60 M=1,N
        DO 60 I=1,NY
        DO 60 J=1,NY
       KI=M*NY +I
       KJ=M*NY +J
 60     XK(KI,KJ)=XK(I,J)
      K1=NY*(NZ-2)
        K2=NY*(NZ-5)
      K3=K1+NY
      K4=K2+NY
```

```
      K5=K4+NY
      K6=K5+NY
      K7=K6+NY
        DO 70 I=1,NY
        XK(I,I)=XK(I,I) + 5.*A12/2.
         XK(I,I+NY)= -8.*A12/3.
         XK(I,I+2*NY)= A12/6.
        XK(I+NY,I+NY)=XK(I+NY,I+NY) + 31.*A12/12.
        XK(NY+I,I)= -4.*A12/3.
        XK(NY+I,2*NY+I) = XK(NY + I,I)
        XK(NY +I,3*NY + I) =A12/12.
       JJ=K1 + I
       XK(JJ,K4+I)=A12/12.
        XK(JJ,K5+I)=-4.*A12/3.
       XK(JJ,K6+I)=XK(JJ,K6+I) + 31.*A12/12.
       XK(JJ,K7+I)= -4.*A12/3.
        K60 =K3 + I
        XK(K60,K4+I)=A12/12.
        XK(K60,K5+I)=-A12/3.
        XK(K60,K6+I)= -17.*A12/12.
        XK(K60,K7+I)=XK(K60,K7+I)  + 5.*A12/3.
70      CONTINUE
      K1=2*NY+1
      K2=(NZ-2)*NY
      J=0
        DO 80 I=K1,K2
        J=J+1
        XK(I,J)=A12/12.
        XK(I,J+NY)=-4.*A12/3.
        XK(I,J+3*NY)= -4.*A12/3.
        XK(I,J+4*NY)=A12/12.
        XK(I,J+2*NY)=XK(I,J+2*NY) + 5.*A12/2.
80      CONTINUE
      RETURN
      END
```

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```
      SUBROUTINE AKY(YK,LY,NZ,NX,HZ,HX,C1,C3)
C
       IMPLICIT REAL*8(A-H,O-Z)
C     A SUBROUTINE TO CALCULATE KY MATRIX
C
C
      DIMENSION YK(LY,1)
C
      A1=C3/(HZ*HZ)
      A2=C1*A1/C3
      A11=C3/(HX*HX)
      A12=C1*A11/C3
C
C     INIALIZE YK
C
      K1=NZ*NX
        DO 10 I=1,K1
        DO 10 J=1,K1
 10     YK(I,J)=0.0
C
C     CALCULATWE ELEMENTS TO FORM K(1,1)
C
       YK(1,1)=5.*A2/2.
       YK(1,2)=-8.*A2/3.
        YK(1,3)=A2/6.
      YK(2,1)= -4.*A2/3.
      YK(2,2)= 31.*A2/12.
      YK(2,3)= YK(2,1)
      YK(2,4)=A2/12.
      YK(3,1)= YK(2,4)
      YK(3,2)=YK(2,1)
       YK(3,3)=5.*A2/2.
      YK(3,4)= YK(2,1)
      YK(3,5) =YK(2,4)
```

```
       YK(NZ-1,NZ-3)=A2/12.
       YK(NZ-1,NZ-2)= -4.*A2/3.
        YK(NZ-1,NZ-1)=31.*A2/12.
        YK(NZ-1,NZ)= -4.*A2/3.
       YK(NZ,NZ-3)=A2/12.
       YK(NZ,NZ-2)=-A2/3.
      YK(NZ,NZ-1)= -17.*A2/12.
       YK(NZ,NZ)=5.*A2/3.
C
C     FILLING THE MIDDLE ROWS OF K(1,1)
C
      K1=NZ-5
      K2=2
      K3=6
      K4=0
      IF(K1.EQ.0) GO TO 40
        DO 20 I=1,K1
        DO 30 J=K2,K3
        K4=K4+1
        YK(I+3,J)= YK(3,K4)
 30     CONTINUE
        K2=K2+1
        K3=K3+1
        K4=0
 20     CONTINUE
C
 40   DO 90 I=1,NZ
 90   WRITE(6,101)(YK(I,J),J=1,NZ)
C
 101  FORMAT(1X,12F10.4)
C
C     FILL ALL THE MATRIX IN DIAGONAL
C
      N=NX-1
```

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```
      DO 60 M=1,N
         DO 60 I=1,NZ
            DO 60 J=1,NZ
      KI=M*NZ + I
      KJ=M*NZ + J
              YK(KI,KJ)=YK(I,J)
 60     CONTINUE
C
C
C     COMPLETE THE FIRST TWO AND LAST TWO BLOCKS OF KY MATRIX
      K1= (NX-2)*NZ
      K2= (NX-5)*NZ
      K3=K1+NZ
      K4=K2+NZ
      K5=K4+NZ
      K6=K5+NZ
      K7=K6+NZ
        DO 70 I=1,NZ
       YK(I,I)=YK(I,I) + 5.*A12/3.
       YK(I,I+NZ)=-17.*A12/12
       YK(I,I + 2*NZ)=-A12/3.
       YK(I,I+3*NZ)=A12/12.
         JJ=I + NZ
         YK(JJ,I)= -4.*A12/3.
         YK(JJ,JJ)=YK(JJ,JJ) + 31.*A12/12.
         YK(JJ,JJ+NZ)= -4.*A12/3.
         YK(JJ,JJ+2*NZ)= A12/12.
         JJ=K1 + I
         YK(JJ,K4 +I)=A12/12.
         YK(JJ,K5+I)= -4.*A12/3.
         YK(JJ,K6+I)=YK(JJ,K6+I) + 31.*A12/12.
         YK(JJ,K7+I)=-4.*A12/3.
       YK(K3+I,K4+I)=A12/12.
       YK(K3+I,K5+I)=-A12/3.
```

```
      YK(K3+I,K6+I)=-17.*A12/12.
      YK(K3+I,K7+I)=YK(K3+I,K7+I) + 5.*A12/3.
   70  CONTINUE
C
C     FILLING THE NX*(NZ-4) ROWS IN THE MIDDLE OF KY MATRIX
C
      K1=2*NZ+1
      K2=(NX-2)*NZ
      J=0
       DO 80 I=K1,K2
       J=J+1
       YK(I,J)=A12/12.
       YK(I,J+NZ)= -4.*A12/3.
       YK(I,J+2*NZ)= YK(I,J+2*NZ) + 5.*A12/2.
       YK(I,J+3*NZ)= -4.*A12/3.
       YK(I,J+4*NZ)= A12/12.
   80  CONTINUE
      RETURN
       END
```

```
      SUBROUTINE AKZ(ZK,LZ,NX,NY,NXC,HX,HY,C1,C3)
C      A SUBROUTINE TO CALCULATE KZ MATRIX
C
C
C
       IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ZK(LZ,1)
C
      A1= C3/(HX*HX)
      A2=C1*A1/C3
      A11= C3/(HY*HY)
      A12= C1*A11/C3
C
C      INIALIZE ZK
      K1=NX*NY
        DO 10 I=1,K1
        DO 10 J=1,K1
        ZK(I,J)=0.0
 10     CONTINUE
C
C     CALCULATE ELEMENTS TO FORM K(1,U)
C
       TEMP1=A2/12.
       TEMP2=-4.*A2/3.
       ZK(1,1)=5.*A2/3.
       ZK(1,2)=-17.*A2/12.
       ZK(1,3)=-A2/3.
       ZK(1,4)=TEMP1
       ZK(2,1)=TEMP2
       ZK(2,2)=31.*A2/12.
       ZK(2,3)=TEMP2
       ZK(2,4)=TEMP1
       ZK(3,1)=TEMP1
       ZK(3,2)=TEMP2
```

```
      ZK(3,3)=5.*A2/2.
      ZK(3,4)=TEMP2
      ZK(3,5)=TEMP1
C
C     FILLING THE MIDDLE ROWS OF K(1,1)
C
      K1= NX-5
      K2=2
      K3=6
      K4=0
      IF(K1.EQ.0) GO TO 40
        DO 20 I=1,K1
        DO 30 J=K2,K3
        K4=K4 + 1
        ZK(I+3,J)=ZK(3,K4)
 30     CONTINUE
      K2=K2 + 1
      K3= K3 + 1
      K4=0
 20     CONTINUE
C
C     FILL THE LAST TWO,LINES
C
 40      DO 50 I=1,2
          DO 50 J=1,5
      ZK(NX-2+I,NX-5+J)=ZK(3-I,6-J)
50      CONTINUE
 190  FORMAT(1X,12F10.4)
C
C    FILL ALL SUBMATRICES IN THE DIAGONAL
      N=NY-1
      DO 60 M=1,N
        DO 60 I=1,NX
          DO 60 J=1,NX
```

```
        KI=M*NX + I
        KJ=M*NX + J
           ZK(KI,KJ)=ZK(I,J)
 60    CONTINUE
C
C      FILL FIRST TWO ROWS OF ZK TILL NXC
C
       DO 100 I=1,NXC
        ZK(I,I)=ZK(I,I) + 5.*A12/3.
         ZK(I,I+NX)= -17.*A12/12.
          ZK(I,I+2*NX)= -A12/3.
          ZK(I,I+3*NX)=A12/12.
        K10=I + NX
        ZK(K10,I)= -4*A12/3.
        ZK(K10,K10)=ZK(K10,K10) + 31.*A12/12.
        ZK(K10,I+2*NX)= -4.*A12/3.
        ZK(K10,I+3*NX)=A12/12.
 100   CONTINUE
       K1=NXC+NX
       K2=K1+NX
       K3=K2+NX
       K4 = NX- NXC
       DO 200 I=1,K4
       K10=NXC + I
       ZK(K10,K10)= ZK(K10,K10) + 5.*A12/2.
       ZK(K10,K1+I) = -8.*A12/3.
       ZK(K10,K2+I)= A12/6.
       K11=K10 + NX
       ZK(K11,K10) = -4.*A12/3.
       ZK(K11,K1+I) = ZK(K11,K1 + I) + 31.*A12/12.
      ZK(K11,K2 + I) = -4.*A12/3.
       ZK(K11,K3 + I)= A12/12.
 200   CONTINUE
C
```

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```
C         FILL THE REMAINING MATRIX LEAVING LAST TWO BLOCKS
C
       K1= 2*NX + 1
       K2 = (NY-2)*NX
       J=0
         DO 80 I=K1,K2
         J=J+1
         ZK(I,J)=A12/12.
         ZK(I,J+NX) = -4.*A12/3.
         ZK(I,J+2*NX)=ZK(I,J+2*NX) + 5.*A12/2.
         ZK(I,J+3*NX)= -4.*A12/3.
         ZK(I,J+4*NX) = A12/12.
 80      CONTINUE
C
C   FILL THE LAST TWO BLOCKS
C
       K1=(NY-2)*NX
       K2=(NY-5)*NX
       K3=K1 + NX
       K4=K2 + NX
       K5=K4 + NX
       K6=K5 + NX
       K7=K6 + NX
         DO 70 I =1,NX
         K10 = K1 + I
        ZK(K10,K4+I)=A12/12.
        ZK(K10,K5+I)= -4.*A12/3.
        ZK(K10,K6+I)=ZK(K10,K6+I) + 31.*A12/12.
        ZK(K10,K7+I)= -4.*A12/3.
         K11 = K3 +I
         ZK(K11,K4+I)=A12/12.
         ZK(K11,K5+I)= -A12/3.
         ZK(K11,K6+I)= -17.*A12/12.
          ZK(K11,K7+I)=ZK(K11,K7+I) + 5.*A12/3.
```

```
 70       CONTINUE
        RETURN
        END
          SUBROUTINE INPUT(LY,ID,NFIX,NFREE)
          IMPLICIT REAL*8(A-H,O-Z)
          DIMENSION ID(1)
C
C          SUBROUTINE TO GENERATE IDENTIFICATION ARRAY
C          AND TO CALCULATE NO. OF FREE NODES
C
 40       READ(5,100)NODE1,ID(NODE1),KN
          IF(KN.EQ.0) GO TO 30
          READ(5,100)NODE2,ID(NODE2)
          K1=NODE1 + 1
          K2=NODE2 - 1
          DO 20 I=K1,K2
          ID(I)=ID(NODE1)
 20       CONTINUE
 30       IF(NODE1.EQ.LY.OR.NODE2.EQ.LY) GO TO 50
           GO TO 40
 50        WRITE(6,200)(ID(I),I=1,LY)
 100      FORMAT(3I5)
 200      FORMAT(10I5)
C
C         COUNTING THE NO. OF FREE AND FIXED NODES
C
          NFREE=0
          DO 60 I=1,LY
          IF(ID(I).EQ.1)NFREE=NFREE + 1
 60        CONTINUE
          NFIX=LY - NFREE
          WRITE(6,200)NFIX,NFREE
          RETURN
          END
```

```
      SUBROUTINE DTHETA(A,N,HI,B)
      IMPLICIT REAL*16(A-H,O-Z)
      DIMENSION A(N,1),B(N,1)
C     A SUBROUTINE TO FORM THETA
C     A IS THE INPUT MATRIX
C     N IS THE ORDER OF A MATRIX
C     HI IS THE MINOR NODAL SPACING
C     B IS (I-HI*HI/4*K)
      H1=-HI*HI/4.
      DO 20 J=1,N
      DO 10 I=1,N
      B(I,J)=H1*A(I,J)
      IF(I.EQ.J)B(I,J)=1.+B(I,J)
 10   CONTINUE
 20   CONTINUE
      RETURN
      END
      SUBROUTINE THETA(A,N,HI,B)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N,1),B(N,1)
C     A SUBROUTINE TO FORM THETA
C     A IS THE INPUT MATRIX
C     N IS THE ORDER OF A MATRIX
C     HI IS THE MINOR NODAL SPACING
C     B IS (I-HI*HI/4*K)
      H1=-HI*HI/4.
      DO 20 J=1,N
      DO 10 I=1,N
      B(I,J)=H1*A(I,J)
      IF(I.EQ.J)B(I,J)=1.+B(I,J)
 10   CONTINUE
 20   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DFORML(A,B,N,M,HI,C)
      IMPLICIT REAL*16(A-H,O-Z)
      DIMENSION A(N,1),B(N,1),C(M,1)
C     A IS KX OR KY OR KZ MATRIX
C     B IS THETA INVERSE MATRIX
C     N IS THE ORDRE OF A AND B MATRIX
C     M IS THE ORDER OF L MATRIX
C     HI IS THE MINOR NODAL SPACING
      DO 20 J=1,N
        JJ=J+N
      DO 10 I=1,N
      II=I+N
      C(I,J)=2.*B(I,J)
      C(I,JJ)=HI*B(I,J)
      C(II,JJ)=C(I,J)
      IF(J.NE.I)GO TO 10
      C(I,J)=C(I,J)-1.0
      C(II,JJ)=C(I,J)
 10   CONTINUE
  20         CONTINUE
      DO 40 I=1,N
      II=I+N
      DO 40 J=1,N
       D=0.0
      DO 50 K=1,N
50     D=D+HI*B(I,K)*A(K,J)
 40          C(II,J)=D
      RETURN
      END
```

```
      SUBROUTINE FORML(A,B,N,M,HI,C)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N,1),B(N,1),C(M,1)
C     A IS KX OR KY OR KZ MATRIX
C     B IS THETA INVERSE MATRIX
C     N IS THE ORDRE OF A AND B MATRIX
C     M IS THE ORDER OF L MATRIX
C     HI IS THE MINOR NODAL SPACING
      DO 20 J=1,N
        JJ=J+N
      DO 10 I=1,N
      II=I+N
      C(I,J)=2.*B(I,J)
      C(I,JJ)=HI*B(I,J)
      C(II,JJ)=C(I,J)
      IF(J.NE.I)GO TO 10
      C(I,J)=C(I,J)-1.0
      C(II,JJ)=C(I,J)
 10   CONTINUE
 20          CONTINUE
      DO 40 I=1,N
      II=I+N
      DO 40 J=1,N
       D=0.0
      DO 50 K=1,N
50     D=D+HI*B(I,K)*A(K,J)
 40          C(II,J)=D
      RETURN
      END
```

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```
      SUBROUTINE DMULT(A,B,N,TEMP)
      IMPLICIT REAL*16(A-H,O-Z)
      DIMENSION A(N,1),B(N,1),TEMP(1)
C
C     A IS A TRANSPOSE MATRIX OF XLN
C     B IS XLL MATRIX
C
      DO 40 I=1,N
      DO 10 J=1,N
      D=0.0
      DO 20 K=1,N
 20   D=D + A(K,I)*B(K,J)
 10   TEMP(J)=D
      DO 30 L=1,N
 30   A(L,I)=TEMP(L)
 40   CONTINUE
      RETURN
      END
      SUBROUTINE MULT1(A,B,N,TEMP)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N,1),B(N,1),TEMP(1)
C
C     A IS A TRANSPOSE MATRIX OF XLN
C     B IS XLL MATRIX
C
      DO 40 I=1,N
      DO 10 J=1,N
      D=0.0
      DO 20 K=1,N
 20   D=D + A(K,I)*B(K,J)
 10   TEMP(J)=D
      DO 30 L=1,N
 30   A(L,I)=TEMP(L)
 40   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DTRANS(A,M,N,MN,MOVE,IWRK,IOK)
C      A IS A ONE-DIMENSIONAL ARRAY OF LENGTH MN=M*N,WHICH
C      CONTAIN THE M*N MATRIX TO BE TRANSPOSED(STORED
C      COLUMNWISE).MOVE IS A ONE-DIMENSIONAL ARRAY OF LENGTH IWRK
C     USED TO STORE INFORMATION TO SPEED UP THE PROCESS.THE
C      VALUE IWRK=(M+N)/2 IS RECOMMENDED.IOK INDICATES THE
C     SUCCESS OR FAILURE OF THE ROUTINE.
C     NORMAL RETURN IOK=0
C     ERRORS              IOK=-1,MN NOT EQUAL TO M*N.
C                         IOK=-2,IWRK NEGATIVE OR ZERO
C                         IOK.GT.0,(SHOULD NEVER OCCUR).IN THIS CASE
C     WE SET IOK EQUAL TO THE FINAL VALUE OF I WHEN THE SEARCH
C     IS COMPLETED BUT SOME LOOPS HAVE NOT BEEN MOVED.
      IMPLICIT REAL*16(A-H,O-Z)
      DIMENSION A(1),MOVE(1)
C
C     CHECK ARGUMENTS AND INITIALIZE
C
      IF(M.LT.2.OR.N.LT.2)GO TO 60
      IF(MN.NE.M*N)GO TO 92
      IF(IWRK.LT.1)GO TO 93
      IF(M.EQ.N)GO TO 70
      NCOUNT=2
      M2=M -2
      DO 10 I=1,IWRK
 10   MOVE(I)=0
      IF(M2.LT.1)GO TO 12
C
C     COUNT NUMBER,NCOUNT,OF SINGLE POINTS.
C
      DO 11 IA=1,M2
      IB=IA*(N-1)/(M-1)
      IF(IA*(N-1).NE.IB*(M-1))GO TO 11
      NCOUNT=NCOUNT + 1
```

```
      I=IA*N+IB
      IF(I.GT.IWRK) GO TO 11
      MOVE(I)=1
 11   CONTINUE
C
C     SET INITIAL VALUES FOR SEARCH
C
 12   K=MN-1
      KMI=K-1
      MAX=MN
      I=1
C
C    AT LEAST ONE LOOP MUST BE RE-ARRANGED.
C
      GO TO 30
C
C     SEARCH FOR LOOPS TO REARRANGE.
C
 20   MAX=K-I
      I=I+1
      KMI=K-I
      IF(I.GT.MAX)GO TO 90
      IF(I.GT.IWRK)GO TO 21
      IF(MOVE(I).LT.1)GO TO 30
      GO TO 20
 21    I2=M*I-K*(I/N)
      IF(I2.LE.I.OR.I2.GE.MAX)GO TO 20
 22    I2=M*I2-K*(I2/N)
      IF(I2.GT.I.AND.I2.LT.MAX)GO TO 22
      IF(I2.NE.I) GO TO 20
C
C     REARRANGE ELEMENTS OF A LOOP
C
 30   I1=I
```

```
 31     B=A(I1+1)
 32     I2=M*I1-K*(I1/N)
       IF(I1.LE.IWRK)MOVE(I1)=2
 33     NCOUNT=NCOUNT+1
       IF(I2.EQ.I.OR.I2.GE.KMI)GO TO 35
 34    A(I1+1)=A(I2+1)
       I1=I2
       GO TO 32
 35    IF(MAX.EQ.KMI.OR.I2.EQ.I)GO TO 41
       MAX=KMI
       GO TO 34
C
C     TEST FOR SYMMETRIC PAIR OF LOOPS
C
 41    A(I1+1)=B
       IF(NCOUNT.GE.MN)GO TO 60
       IF(I2.EQ.MAX.OR.MAX.EQ.KMI)GO TO 20
       MAX=KMI
       I1=MAX
       GO TO 31
C
C     NORMAL RETURN
C
 60    IOK=0
      RETURN
C
C     IF MATRIX IS SQUARE,EXCHANGE ELEMENTS A(I,J)AND A(J,I)
C
 70    N1=N-1
       DO 71 I=1,N1
       J1=I + 1
       DO 71 J=J1,N
       I1=I+(J-1)*N
       I2=J+(I-1)*M
```

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```
      B=A(I1)
      A(I1)=A(I2)
      A(I2)=B
 71   CONTINUE
      GO TO 60
C
C     ERROR RETURN
C
 90    IOK=I
 91   RETURN
 92    IOK=-1
       GO TO 91
 93    IOK=-2
      GO TO 91
C
      END
      SUBROUTINE TRANS(A,M,N,MN,MOVE,IWRK[illegible]
C      A IS A ONE-DIMENSIONAL ARRAY OF LE[illegible]N=M*N,WHICH
C      CONTAIN THE M*N MATRIX TO BE TRANS[illegible]ED(STORED
C      COLUMNWISE).MOVE IS A ONE-DIMENSIONAL ARRAY OF LENGTH IWRK
C     USED TO STORE INFORMATION TO SPEED UP THE PROCESS.THE
C      VALUE IWRK=(M+N)/2 IS RECOMMENDED.IOK INDICATES THE
C     SUCCESS OR FAILURE OF THE ROUTINE.
C     NORMAL RETURN IOK=0
C     ERRORS          IOK=-1,MN NOT EQUAL TO M*N.
C                     IOK=-2,IWRK NEGATIVE OR ZERO
C                     IOK.GT.0,(SHOULD NEVER OCCUR).IN THIS CASE
C     WE SET IOK EQUAL TO THE FINAL VALUE OF I WHEN THE SEARCH
C     IS COMPLETED BUT SOME LOOPS HAVE NOT BEEN MOVED.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),MOVE(1)
C
C     CHECK ARGUMENTS AND INITIALIZE
C
```

```
      IF(M.LT.2.OR.N.LT.2)GO TO 60
      IF(MN.NE.M*N)GO TO 92
      IF(IWRK.LT.1)GO TO 93
      IF(M.EQ.N)GO TO 70
      NCOUNT=2
      M2=M -2
      DO 10 I=1,IWRK
 10   MOVE(I)=0
      IF(M2.LT.1)GO TO 12
C
C     COUNT NUMBER,NCOUNT,OF SINGLE POINTS.
C
      DO 11 IA=1,M2
      IB=IA*(N-1)/(M-1)
      IF(IA*(N-1).NE.IB*(M-1))GO TO 11
      NCOUNT=NCOUNT + 1
      I=IA*N+IB
      IF(I.GT.IWRK) GO TO 11
      MOVE(I)=1
 11   CONTINUE
C
C     SET INITIAL VALUES FOR SEARCH
C
 12   K=MN-1
      KMI=K-1
      MAX=MN
      I=1
C
C    AT LEAST ONE LOOP MUST BE RE-ARRANGED.
C
      GO TO 30
C
C     SEARCH FOR LOOPS TO REARRANGE.
C
```

```
 20      MAX=K-I
         I=I+1
         KMI=K-I
         IF(I.GT.MAX)GO TO 90
         IF(I.GT.IWRK)GO TO 21
         IF(MOVE(I).LT.1)GO TO 30
         GO TO 20
 21       I2=M*I-K*(I/N)
         IF(I2.LE.I.OR.I2.GE.MAX)GO TO 20
 22       I2=M*I2-K*(I2/N)
         IF(I2.GT.I.AND.I2.LT.MAX)GO TO 22
         IF(I2.NE.I) GO TO 20
C
C        REARRANGE ELEMENTS OF A LOOP
C
 30      I1=I
 31       B=A(I1+1)
 32       I2=M*I1-K*(I1/N)
         IF(I1.LE.IWRK)MOVE(I1)=2
 33       NCOUNT=NCOUNT+1
         IF(I2.EQ.I.OR.I2.GE.KMI)GO TO 35
 34      A(I1+1)=A(I2+1)
         I1=I2
         GO TO 32
 35      IF(MAX.EQ.KMI.OR.I2.EQ.I)GO TO 41
         MAX=KMI
         GO TO 34
C
C       TEST FOR SYMMETRIC PAIR OF LOOPS
C
 41      A(I1+1)=B
         IF(NCOUNT.GE.MN)GO TO 60
         IF(I2.EQ.MAX.OR.MAX.EQ.KMI)GO TO 20
         MAX=KMI
```

```
            I1=MAX
            GO TO 31
C
C          NORMAL RETURN
C
 60         IOK=0
           RETURN
C
C          IF MATRIX IS SQUARE,EXCHANGE ELEMENTS A(I,J)AND A(J,I)
C
 70         N1=N-1
            DO 71 I=1,N1
            J1=I + 1
            DO 71 J=J1,N
            I1=I+(J-1)*N
            I2=J+(I-1)*M
            B=A(I1)
            A(I1)=A(I2)
            A(I2)=B
 71         CONTINUE
            GO TO 60
C
C           ERROR RETURN
C
 90          IOK=I
 91         RETURN
 92          IOK=-1
             GO TO 91
 93          IOK=-2
            GO TO 91
C
            END
```

```
      SUBROUTINE DMINV2(A,N,L,M)
     IMPLICIT REAL*16(A-H,O-Z)
        DIMENSION A(N,1),L(1),M(1)
1000    FORMAT(1X,10E12.4)
        FAC=1.0
30      FAC=10.*FAC
        DIV=A(1,1)/FAC
        DIV1=QABS(DIV)
         IF(DIV1.GT.10.)GO TO 30
          DO 40 I=1,N
          DO 40 J=1,N
          A(I,J)=A(I,J)/FAC
40         CONTINUE
          CALL DMINV(A,N,D,L,M)
5000      FORMAT(1X,'THE DETERMINANT ',E12.4/)
         DO 50 I=1,N
          DO 50 J=1,N
50        A(I,J)=A(I,J)/FAC
C
          RETURN
           END
      SUBROUTINE DMINV(A,N,D,L,M)
      DIMENSION A(1),L(1),M(1)
      REAL*16 A,D,BIGA,HOLD
C
C     SEARCH FOR LARGEST ELEMENT
C
      D=1.0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
```

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```
      BIGA=A(KK)
      DO 20   =K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
 10   IF(QABS(BIGA)-QABS(A(IJ))) 15,20,20
 15   BIGA=A(IJ)
      L(K)=I
      M(K)=J
 20   CONTINUE
C
C     INTERCHANGE ROWS
C
      J=L(K)
      IF(J-K) 35,35,25
 25   KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
 30   A(JI) =HOLD
C
C     INTERCHANGE COLUMN
C
 35   I=M(K)
      IF(I-K) 45,45,38
 38   JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
 40   A(JI) =HOLD
```

```
C
C       DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
C       CONTAINED IN BIGA)
C
   45   IF(BIGA) 48,46,48
   46   D=0.0
        RETURN
   48   DO 55 I=1,N
        IF(I-K) 50,55,50
   50   IK=NK+I
        A(IK)=A(IK)/(-BIGA)
   55   CONTINUE
C
C       REDUCE MATRIX
C
        DO 65 I=1,N
        IK=NK+I
        HOLD=A(IK)
        IJ=I-N
        DO 65 J=1,N
        IJ=IJ+N
        IF(I-K) 60,65,60
   60   IF(J-K) 62,65,62
   62   KJ=IJ-I+K
        A(IJ)=HOLD*A(KJ)+A(IJ)
   65   CONTINUE
C
C       DIVIDE ROW BY PIVOT
C
        KJ=K-N
        DO 75 J=1,N
        KJ=KJ+N
        IF(J-K) 70,75,70
   70   A(KJ)=A(KJ)/BIGA
   75   CONTINUE
```

```
C
C       PRODUCT OF PIVOT
C
C        D=D*BIGA
1000     FORMAT(1X,E15.8)
C
C       REPLACE PIVOT BY RECIPROCAL
C
        A(KK)=1.0/BIGA
 80     CONTINUE
C
C       FINAL ROW AND COLUMN INTERCHANGE
C
        K=N
 100    K=(K-1)
        IF(K) 150,150,105
 105    I=L(K)
        IF(I-K) 120,120,108
 108    JQ=N*(K-1)
        JR=N*(I-1)
        DO 110 J=1,N
        JK=JQ+J
        HOLD=A(JK)
        JI=JR+J
        A(JK)=-A(JI)
 110    A(JI) =HOLD
 120    J=M(K)
        IF(J-K) 100,100,125
 125    KI=K-N
        DO 130 I=1,N
        KI=KI+N
        HOLD=A(KI)
        JI=KI-K+J
```

```
      A(KI)=-A(JI)
130   A(JI) =HOLD
      GO TO 100
150   RETURN
      END
         SUBROUTINE MINV1(A,N,L,M)
       IMPLICIT REAL*8(A-H,O-Z)
          DIMENSION A(N,1),L(1),M(1)
1000      FORMAT(1X,10E12.4)
          FAC=1.0
30        FAC=10.*FAC
          DIV=A(1,1)/FAC
          DIV1=DABS(DIV)
           IF(DIV1.GT.10.)GO TO 30
            DO 40 I=1,N
            DO 40 J=1,N
            A(I,J)=A(I,J)/FAC
40           CONTINUE
            CALL MINV(A,N,D,L,M)
5000        FORMAT(1X,'THE DETERMINANT ',E12.4/)
           DO 50 I=1,N
            DO 50 J=1,N
50          A(I,J)=A(I,J)/FAC
C
            RETURN
             END
      SUBROUTINE MINV(A,N,D,L,M)
      DIMENSION A(1),L(1),M(1)
      DOUBLE PRECISION A,D,BIGA,HOLD,DABS
C
C     SEARCH FOR LARGEST ELEMENT
C
      D=1.0
      NK=-N
```

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```
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
 10   IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
 15   BIGA=A(IJ)
      L(K)=I
      M(K)=J
 20   CONTINUE
C
C     INTERCHANGE ROWS
C
      J=L(K)
      IF(J-K) 35,35,25
 25   KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
 30   A(JI) =HOLD
C
C     INTERCHANGE COLUMN
C
 35   I=M(K)
      IF(I-K) 45,45,38
 38   JP=N*(I-1)
      DO 40 J=1,N
```

```
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
 40   A(JI) =HOLD
C
C     DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
C     CONTAINED IN BIGA)
C
 45   IF(BIGA) 48,46,48
 46   D=0.0
      RETURN
 48   DO 55 I=1,N
      IF(I-K) 50,55,50
 50   IK=NK+I
      A(IK)=A(IK)/(-BIGA)
 55   CONTINUE
C
C     REDUCE MATRIX
C
      DO 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
 60   IF(J-K) 62,65,62
 62   KJ=IJ-I+K
      A(IJ)=HOLD*A(KJ)+A(IJ)
 65   CONTINUE
C
C     DIVIDE ROW BY PIVOT
C
```

C-4

```
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
 70   A(KJ)=A(KJ)/BIGA
 75   CONTINUE
C
C     PRODUCT OF PIVOT
C
      D=D*BIGA
C
C     REPLACE PIVOT BY RECIPROCAL
C
      A(KK)=1.0/BIGA
 80   CONTINUE
C
C     FINAL ROW AND COLUMN INTERCHANGE
C
      K=N
 100  K=(K-1)
      IF(K) 150,150,105
 105  I=L(K)
      IF(I-K) 120,120,108
 108  JQ=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
 110  A(JI) =HOLD
 120  J=M(K)
      IF(J-K) 100,100,125
 125  KI=K-N
```

```
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
130   A(JI) =HOLD
      GO TO 100
150   RETURN
      END
      SUBROUTINE TRAP(A,B,N,M)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N,1),B(M,1)
C     A IS THE INPUT MATRIX
C     B IS THE OUTPUT MATRIX
      DO 10 I=1,N
      DO 10 J=1,M
10    B(J,I)=A(I,J)
      RETURN
      END
      SUBROUTINE DTRAP(A,B,N,M)
      IMPLICIT REAL*16(A-H,O-Z)
      DIMENSION A(N,1),B(M,1)
C     A IS THE INPUT MATRIX
C     B IS THE OUTPUT MATRIX
      DO 10 I=1,N
      DO 10 J=1,M
10    B(J,I)=A(I,J)
      RETURN
      END
      SUBROUTINE INIT(A,N,M)
      DOUBLE PRECISION A
      DIMENSION A(N,1)
       DO 10 J=1,M
        DO 10 I=1,N
10    A(I,J)=0.0
```

```
      RETURN
      END
       SUBROUTINE FORMR(A,B,N,L,N1,L1,L2,L3,L4,C2,AK,IDIM1)
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION AK(IDIM1,1),A(L3,1),B(L4,1)
C
C      A IS V  W  OR  U
C       B  IS W  U  OR  V
C  ORDER OF VARIABLES.........U  V  W
C       N  IS  NX  NY  OR  NZ
C       L  IS  LX  LY  OR  LZ
C       N1  IS  NY  NZ  OR  NX
C  L1  IS  LY  LZ  OR  LX
C       L2  IS  LZ  LX  OR  LY
C       L3  IS  LY1  LZ1  OR  LX1
C       L4  IS  LZ1  LX1  OR  LY1
C
       II=L1 +1
       JJ=0
       DO 10 I=1,N
       MM=L2 + I
       KK=1
       DO 20 J=1,L
       JJ=JJ + 1
       AK(J,I)=C2*(A(II,JJ) + B(MM,KK))
       MM=MM + N
        IF(JJ.NE.N1) GO TO 20
       JJ=0
       KK=KK + 1
       MM=L2 + I
       II=II + 1
 20    CONTINUE
 10     CONTINUE
       RETURN
       END
```

```
      SUBROUTINE DORMRF(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2
      IMPLICIT REAL*8(A-H,O-W),REAL*16(X),REAL*8(Y,Z)
      DIMENSION XK1(LX,1),XLL(LX1,1),AK(IDIM1,1),B(IDIM2,1)
C
C       THIS SUBROUTINE IS FORMED USING A PART OF OLD SUBROUTINE
C      ALL THE ARGUMENTS ARE REPLACED BY RESPECTIVE VALUES FOR Y AND
C
C
C     INTERPOLATION OF VALUES
C
      X0=0.0
      X1=X0 + HX
      JJ=3
      X=X0
      K1=NX - 2
      K2=NMX - 1
      DO 80 J=1,K1
      DO 90 I=1,LX
      B(I,1)=AK(I,J)
      B(I,2)=(AK(I,J+1) - AK(I,J))/HX
      B(I,3)=(AK(I,J+2)-AK(I,J+1)-B(I,2)*HX)/(2.*HX*HX)
 90   CONTINUE
      DO 101 K=1,K2
      X=X + HMX
      AA=X - X0
      BB=X - X1
      JJ=JJ + 1
      DO 101 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
 101  CONTINUE
      X0=X1
      X1=X1 + HX
      X=X0
 80   CONTINUE
```

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```
         X=XO + HX
C
C       INTERPOLATION FOR LAST INTERVAL
        DO 110 K=1,K2
        X=X + HMX
        JJ=JJ + 1
        AA=X - XO
        BB=X - X1
        DO 110 I=1,LX
        B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
 110    CONTINUE
C
C       GENERATION OF RI MATR)X INCLUSING MINOR AND MAJOR NODES
C
        K1=(NX-1)*NMX + 1
        DO 120 I=1,LX
        B(I,K1)=AK(I,1)
 120    CONTINUE
        JJ=3
        DO 130 J=2,NX
        DO 140 JA=1,K2
        K1=K1 + 1
        JJ=JJ + 1
        DO 150 I=1,LX
        B(I,K1)=B(I,JJ)
 150    CONTINUE
 140    CONTINUE
        K1=K1 + 1
        DO 160 I=1,LX
        B(I,K1)=AK(I,J)
 160    CONTINUE
 130    CONTINUE
 8000   FORMAT(1X,'THE RX VECTORS ARE'/)
C       CALCULATION OF RI-RI-1
```

```
      K2=(NX-1)*NMX
      DO 170 J=1,K2
      K3=K2 + J
      K4=K3 + 1
      DO 180 I=1,LX
      B(I,K3)=B(I,K4) - B(I,K3)
 180  CONTINUE
 170  CONTINUE
      ITER=ITER + 1
C     INITIALIZATION OF B
C     CALCULATION OF M
C
       DO 201 I=1,LX
       II=I + LX
       DO 201 J=1,K2
       JJ=K2 + J
       TEMP=0.0
       DO 202 K=1,LX
       TEMP=TEMP + XK1(I,K)*B(K,JJ)
 202   CONTINUE
       B(I,J)=HMX*TEMP/2.
       B(II,J)=TEMP
 201   CONTINUE
C
C     CALCULATION OF FN
C
      K3=K2 + 1
      K4=K3 + 1
      DO 230 I=1,LX1
 230  B(I,K3)=0.0
      DO 240 J=1,K2
      DO 250 I=1,LX1
       TEMP=0.0
       DO 251 K=1,LX1
```

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```
        TEMP=TEMP + XLL(K,I)*B(K,K3)
 251    CONTINUE
 250    B(I,K4)=TEMP
       DO 260 I=1,LX1
       B(I,K3)=B(I,K4) + B(I,J)
 260   CONTINUE
 240   CONTINUE
 3     FORMAT(1X,'THE FN VECTOR IS GIVEN'//)
 7000  FORMAT(1X,10E12.4)
        RETURN
       END
        SUBROUTINE DFORMD(XLL,B,NX,NMX,LX1,U,XV2K,XVNK,IDIM2)
       IMPLICIT REAL*8(A-H,O-W),REAL*16(X),REAL*8(Y,Z)
C
C       THIS PART OF SUBROUTINE IS OBTAINED FROM OLD UDISP
C
        DIMENSION XLL(LX1,1),XV2K(1),XVNK(1),U(LX1,1),B(IDIM2,1)
C      CALCULATION OF ALL NODAL DISPLACENENT
C
       DO 400 I=1,LX1
       XVNK(I)=U(I,1)
 400   CONTINUE
       JJ=NX - 1
       KK=0
       DO 410 I=1,JJ
       NN=I + 1
       DO 420 II=1,NMX
       DO 430 J=1,LX1
        XTEMP=0.0
        DO 431 K=1,LX1
        XTEMP=XTEMP + XLL(K,J)*XVNK(K)
 431    CONTINUE
 430    XV2K(J)=XTEMP
       KK=KK + 1
```

```
      DO 440 K=1,LX1
       XVNK(K)=XV2K(K) + B(K,KK)
 440  CONTINUE
 420  CONTINUE
      DO 460 K=1,LX1
      U(K,NN)=XVNK(K)
 460  CONTINUE
 410  CONTINUE
C
       RETURN
       END
       SUBROUTINE RF(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2,NZ,
     *C1,H)
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION XK1(LX,1),XLL(IDIM2,1),AK(IDIM1,1),B(IDIM2,1)
C     DIMENSION XLL HAS BEEN CHANGED TO ACCOM. LY NE. LZ
C
C         THIS SUBROUTINE IS FORMED USING A PART OF OLD SUBROUTINE
C        ALL THE ARGUMENTS ARE REPLACED BY RESPECTIVE VALUES FOR Y AND
C
C
C     INTERPOLATION OF VALUES
C
      X0=0.0
      X1=X0 + HX
      JJ=3
      X=X0
      K1=NX - 2
      K2=NMX - 1
      DO 80 J=1,K1
      DO 90 I=1,LX
      B(I,1)=AK(I,J)
      B(I,2)=(AK(I,J+1) - AK(I,J))/HX
      B(I,3)=(AK(I,J+2)-AK(I,J+1)-B(I,2)*HX)/(2.*HX*HX)
```

```
 90     CONTINUE
        DO 101 K=1,K2
        X=X + HMX
        AA=X - XO
        BB=X - X1
        JJ=JJ + 1
        DO 101 I=1,LX
        B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
 101    CONTINUE
        XO=X1
        X1=X1 + HX
        X=XO
 80     CONTINUE
           X=XO + HX
C
C       INTERPOLATION FOR LAST INTERVAL
        DO 110 K=1,K2
        X=X + HMX
        JJ=JJ + 1
        AA=X - XO
        BB=X - X1
        DO 110 I=1,LX
        B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
 110    CONTINUE
C
C       GENERATION OF RI MATRIX INCLUSING MINOR AND MAJOR NODES
C
        K1=(NX-1)*NMX + 1
        DO 120 I=1,LX
        B(I,K1)=AK(I,1)
 120    CONTINUE
        JJ=3
        DO 130 J=2,NX
        DO 140 JA=1,K2
```

```
      K1=K1 + 1
      JJ=JJ + 1
      DO 150 I=1,LX
      B(I,K1)=B(I,JJ)
 150  CONTINUE
 140  CONTINUE
      K1=K1 + 1
      DO 160 I=1,LX
      B(I,K1)=AK(I,J)
 160  CONTINUE
 130  CONTINUE
 8009 FORMAT(1X,'THE RX VECTORS ARE'/)
C     CALCULATION OF RI-RI-1
C
      A2=HMX*C1/(12.0*H)
      A1=-11.0*A2
      N10=2*NZ
      SI=HMX/((NX - 1)*HX)
7001  FORMAT(I5)
C
      K2=(NX-1)*NMX
      DO 170 J=1,K2
         DY1=(J - 1)*SI
         DY2=DY1 + SI
         AL1=4.*DY1*(1 - DY1)
         AL2=4.*DY2*(1 - DY2)
         A11=A1*(AL1 + AL2)
         A22=A2*(AL1 + AL2)
      K3=K2 + J
      K4=K3 + 1
      DO 180 I=1,LX
      IF(I.GT.NZ) GO TO 181
      B(I,K3)=B(I,K4) - B(I,K3) + A11
      GO TO 180
```

```
181     IF(I.GT.N10) GO TO 182
        B(I,K3)=B(I,K4) - B(I,K3) + A22
        GO TO 180
182     B(I,K3)=B(I,K4) - B(I,K3)
180     CONTINUE
 170    CONTINUE
C       INITIALIZATION OF B
C       CALCULATION OF M
C
         DO 201 I=1,LX
         II=I + LX
         DO 201 J=1,K2
         JJ=K2 + J
         TEMP=0.0
         DO 202 K=1,LX
         TEMP=TEMP + XK1(I,K)*B(K,JJ)
 202     CONTINUE
         B(I,J)=HMX*TEMP/2.
         B(II,J)=TEMP
 201     CONTINUE
C
C       CALCULATION OF FN
C
        K3=K2 + 1
        K4=K3 + 1
        DO 230 I=1,LX1
 230    B(I,K3)=0.0
        DO 240 J=1,K2
        DO 250 I=1,LX1
         TEMP=0.0
         DO 251 K=1,LX1
         TEMP=TEMP + XLL(K,I)*B(K,K3)
 251     CONTINUE
 250     B(I,K4)=TEMP
```

```
      DO 260 I=1,LX1
      B(I,K3)=B(I,K4) + B(I,J)
 260  CONTINUE
 240  CONTINUE
 3    FORMAT(1X,'THE FN VECTOR IS GIVEN'//)
 7000 FORMAT(1X,10E12.4)
       RETURN
      END

       SUBROUTINE FORMD(XLL,B,NX,NMX,LX1,U,V2K,VNK,IDIM2)
       IMPLICIT REAL*8(A-H,O-Z)
C
C      THIS PART OF SUBROUTINE IS OBTAINED FROM OLD UDISP
C
       DIMENSION XLL(IDIM2,1),V2K(1),VNK(1),U(LX1,1),B(IDIM2,1)
C      DIMEN OF XLL HAS BEE  CHANGED TO ACCOM LY NE. LZ
C     CALCULATION OF ALL NODAL DISPLACENENT
C
      DO 400 I=1,LX1
      VNK(I)=U(I,1)
 400  CONTINUE
      JJ=NX - 1
      KK=0
      DO 410 I=1,JJ
      NN=I + 1
      DO 420 II=1,NMX
      DO 430 J=1,LX1
       TEMP=0.0
       DO 431 K=1,LX1
       TEMP=TEMP + XLL(K,J)*VNK(K)
 431   CONTINUE
 430   V2K(J)=TEMP
      KK=KK + 1
      DO 440 K=1,LX1
       VNK(K)=V2K(K) + B(K,KK)
```

```
 440    CONTINUE
 420    CONTINUE
        DO 460 K=1,LX1
        U(K,NN)=VNK(K)
 460    CONTINUE
 410    CONTINUE
C
         RETURN
              END
              SUBROUTINE FORMRF(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2
              IMPLICIT REAL*8(A-H,O-Z)
              DIMENSION XK1(LX,1),XLL(IDIM2,1),AK(IDIM1,1),B(IDIM2,1)
C           DIMEN XLL HAS BEEN CHANGED TO ACCOM. LY NE.LZ
C
C               THIS SUBROUTINE IS FORMED USING A PART OF OLD SUBROUTINE
C              ALL THE ARGUMENTS ARE REPLACED BY RESPECTIVE VALUES FOR Y AND
C
C
C            INTERPOLATION OF VALUES
C
             X0=0.0
             X1=X0 + HX
             JJ=3
             X=X0
             K1=NX - 2
             K2=NMX - 1
             DO 80 J=1,K1
             DO 90 I=1,LX
             B(I,1)=AK(I,J)
             B(I,2)=(AK(I,J+1) - AK(I,J))/HX
             B(I,3)=(AK(I,J+2)-AK(I,J+1)-B(I,2)*HX)/(2.*HX*HX)
   90        CONTINUE
             DO 101 K=1,K2
             X=X + HMX
```

```
      AA=X - X0
      BB=X - X1
      JJ=JJ + 1
      DO 101 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
 101  CONTINUE
      X0=X1
      X1=X1 + HX
      X=X0
 80   CONTINUE
        X=X0 + HX
C
C     INTERPOLATION FOR LAST INTERVAL
      DO 110 K=1,K2
      X=X + HMX
      JJ=JJ + 1
      AA=X - X0
      BB=X - X1
      DO 110 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
 110  CONTINUE
C
C     GENERATION OF RI MATRIX INCLUSING MINOR AND MAJOR NODES
C
      K1=(NX-1)*NMX + 1
      DO 120 I=1,LX
      B(I,K1)=AK(I,1)
 120  CONTINUE
      JJ=3
      DO 130 J=2,NX
      DO 140 JA=1,K2
      K1=K1 + 1
      JJ=JJ + 1
      DO 150 I=1,LX
```

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```
         B(I,K1)=B(I,JJ)
  150    CONTINUE
  140    CONTINUE
         K1=K1 + 1
         DO 160 I=1,LX
         B(I,K1)=AK(I,J)
  160    CONTINUE
  130    CONTINUE
 8000    FORMAT(1X,'THE RX VECTORS ARE'/)
C        CALCULATION OF RI-RI-1
         K2=(NX-1)*NMX
         DO 170 J=1,K2
         K3=K2 + J
         K4=K3 + 1
         DO 180 I=1,LX
         B(I,K3)=B(I,K4) - B(I,K3)
  180    CONTINUE
  170    CONTINUE
         ITER=ITER + 1
C        INITIALIZATION OF B
C        CALCULATION OF M
C
          DO 201 I=1,LX
          II=I + LX
          DO 201 J=1,K2
          JJ=K2 + J
          TEMP=0.0
          DO 202 K=1,LX
          TEMP=TEMP + XK1(I,K)*B(K,JJ)
  202     CONTINUE
          B(I,J)=HMX*TEMP/2.
          B(II,J)=TEMP
  201     CONTINUE
C
```

```
C       CALCULATION OF FN
C
        K3=K2 + 1
        K4=K3 + 1
        DO 230 I=1,LX1
 230    B(I,K3)=0.0
        DO 240 J=1,K2
        DO 250 I=1,LX1
         TEMP=0.0
         DO 251 K=1,LX1
         TEMP=TEMP + XLL(K,I)*B(K,K3)
 251     CONTINUE
 250     B(I,K4)=TEMP
        DO 260 I=1,LX1
        B(I,K3)=B(I,K4) + B(I,J)
 260    CONTINUE
 240    CONTINUE
 3      FORMAT(1X,'THE FN VECTOR IS GIVEN'//)
 7000   FORMAT(1X,10E12.4)
         RETURN
        END
```

```
      SUBROUTINE PUDISP(U,V,W,AK,XK1,XLL,XLOC1,V2K,VNK,L,M,B,XLOC,FN
     1,EPSX,DEPX,EPSXY,DEPXY,EPSZX,DEPZX,DR,DR1,DR2)
       IMPLICIT REAL*8(A-H,O-W),REAL*16(X),REAL*8(Y,Z)
      COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
      COMMON/BLK2/HX,HY,HZ,HMX,HMY,HMZ,C1,C2,C3,C4
       COMMON/BLK4/IDIM1,IDIM2
         DIMENSION U(LX1,1),V(LY1,1),W(LZ1,1),AK(IDIM1,1),XLL(LX1,1)
         DIMENSION XLOC1(LX,1),VNK(1),L(1),M(1),B(IDIM2,1),V2K(1)
       DIMENSION XK1(LX,1)
      DIMENSION XV2K(98),XVNK(98)
      DIMENSION EPSX(LX,1),DEPX(LX,1)
      DIMENSION EPSXY(LX,1),DEPXY(LX,1),EPSZX(LZ,1),DEPZX(LZ,1)
      DIMENSION DR(IDIM1,1),DR1(IDIM1,1),DR2(IDIM1,1)
C
C        IN THIS SUBROUTINE XLN HAS BEEN TAKEN OUT AND IS REPLACED BY
C        XLOC1(LX,1) WHICH IS ALPHA4
C
         DIMENSION XLOC(LX,1),FN(1)
C      GENERATION OF R VECTORS
C
C
        CALL FORMR(V,W,NX,LX,NY,LY,LZ,LY1,LZ1,C2,AK,IDIM1)
C
C      ADDITION OF PLASTIC TERMS
C
      CC=2.0*C1
      DO 60 I=1,NX
      DO 60 J=1,LX
      AK(J,I)=AK(J,I) + CC*(EPSX(J,I) + DEPX(J,I))
60    CONTINUE
C
      CALL SHER(EPSXY,DEPXY,EPSZX,DEPZX,LX,LZ,NX,NY,NZ
     *,C1,HY,HZ,IDIM1,DR)
C
```

```
C
      C5=C1/(6.*HY)
      C6=11.*C5
     C7=C1/(6.*HZ)
      C8=11.*C7
     N10=NY - 1
     JJ=0
      II=1
      DO 71 I=1,NX
       DO 72 J=1,LX
       JJ=JJ + 1
        IF(JJ.EQ.1)AK(J,I)=AK(J,I) - C6*V(II,JJ)
        IF(JJ.EQ.2)AK(J,I)=AK(J,I) + C5*V(II,JJ-1)
        IF(JJ.EQ.N10)AK(J,I)=AK(J,I) - C5*V(II,JJ+1)
        IF(JJ.NE.NY)GO TO 72
       AK(J,I)=AK(J,I) + C6*V(II,JJ)
       JJ=0
        II=II + 1
 72     CONTINUE
 71     CONTINUE
      N11=(NZ -2)*NY + 1
      N12=(NZ-1)*NY
      N14=NZ-1
      DO 73 I=1,NX
      MM=I
       DO 74 J=N11,N12
        AK(J,I)=AK(J,I) - C7*W(MM,NZ)
        N13=J + NY
        AK(N13,I)=AK(N13,I) + C8*W(MM,NZ)
          MM= MM + NX
 74      CONTINUE
  73     CONTINUE
C
     CALL PDORM(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2,
```

```
     *DR,DR1,DR2)
C
C       GENERATION OF INITIAL CONDITION ON FACE FOUR
C
        II=LY + 1
        JJ=0
         MM=LZ + 1
         KK=1
        DO 280 J=1,LX
        JJ=JJ + 1
       AA=C3*(V(II,JJ) + W(MM,KK))
       BB=CC*(EPSX(J,1) + DEPX(J,1))
       XV2K(J)=AA + BB
         MM=MM + NX
        IF(JJ.NE.NY) GO TO 280
        JJ=0
         KK=KK + 1
         MM=LZ +1
        II=II + 1
 280    CONTINUE
C
C       GENERATION OF INITIAL CONDITION ON FACE ONE
C
        II=LY + (NX-1)*NZ + 1
        JJ=0
         MM=LZ + NX
         KK=1
        DO 310 J=1,LX
        JJ=JJ + 1
       AA=C3*(V(II,JJ) + W(MM,KK))
       BB=CC*(EPSX(J,NX) + DEPX(J,NX))
       XVNK(J)=AA + BB
         MM=MM + NX
        IF(JJ.NE.NY) GO TO 310
```

```
      JJ=0
       KK=KK + 1
       MM=LZ + NX
      II=II + 1
 310  CONTINUE
C
CSET LAST ELEMENT OF VNK AS ZERO
C
         TEMP=XVNK(LX)
         XVNK(LX)=0.0
C
       K3=(NX - 1)*NMX + 1
C
C
             B(LX1,K3)=B(LX,K3)
C
C
C     CALCULATION FOR UNKNOWN PART OF VECTOR U
C
C     INITIALIZATION OF PART OF B TO BE USED AS WORKSPACE
C
       K1=(NX - 1)*NMX + 1
       DO 360 I=1,LX
       II=I + LX
       U(II,1)=XV2K(I)
       XTEM=0.0
       DO 361 K=1,LX
       XTEM=XTEM + XLOC1(I,K)*XV2K(K)
 361   CONTINUE
       XVNK(I)=XVNK(I) - XTEM - B(II,K1)
 360   CONTINUE
C
       DO 380 I=1,LX
       XTEM=0.0
```

```
      DO 381 K=1,LX
      XTEM=XTEM + XLOC(I,K)*XVNK(K)
 381  CONTINUE
      U(I,1)=XTEM
 380  CONTINUE
C
      CALL DFORMD(XLL,B,NX,NMX,LX1,U,XV2K,XVNK,IDIM2)
C
       U(LX1,NX)=TEMP
 5000 FORMAT(1X,'THE VALUE OF DETERMINANT IS',3X,E12.4)
 6000 FORMAT(1X,'THIS STEP IS DONE',3X,I4)
      RETURN
      END
```

```
      SUBROUTINE PCCV(U,V,W,AK,YK1,XLL,B23,V2K,VNK,BK,L,M,B,FN
     1,EPSY,DEPY,R4,EPSYZ,DEPYZ,EPSXY,DEPXY,DR,DR1,DR2)
       IMPLICIT REAL*8(A-H,O-Z)
      COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
      COMMON/BLK2/HX,HY,HZ,HMX,HMY,HMZ,C1,C2,C3,C4
       COMMON/BLK3/NFIX,NFREE,ID(1)
       COMMON/BLK4/IDIM1,IDIM2
         DIMENSION U(LX1,1),V(LY1,1),W(LZ1,1),AK(IDIM1,1),XLL(IDIM2,1)
C        DIMEN OF XLL IS CHANGE TO ACCOM. LY NE.LZ CASE
         DIMENSION B23(LY,1),VNK(1),L(1),M(1),B(IDIM2,1),YK1(LY,1)
       DIMENSION V2K(1)
      DIMENSION EPSY(LY,1),DEPY(LY,1)
      DIMENSION EPSYZ(LY,1),DEPYZ(LY,1),EPSXY(LX,1),DEPXY(LX,1)
      DIMENSION DR(IDIM1,1),DR1(IDIM1,1),DR2(IDIM1,1)
         DIMENSION BK(LY,1),FN(1)
C      GENERATION OF R VECTORS
C
C
       CALL FORMR(W,U,NY,LY,NZ,LZ,LX,LZ1,LX1,C2,AK,IDIM1)
C
C      ADDITION OF PLASTIC TERMS
C
      CC=2.0*C1
      DO 60 I=1,NY
      DO 60 J=1,LY
      AK(J,I)=AK(J,I) + CC*(EPSY(J,I) + DEPY(J,I))
60    CONTINUE
      CALL SHER(EPSYZ,DEPYZ,EPSXY,DEPXY,LY,LX,NY,NZ,NX,
     *C1,HZ,HX,IDIM1,DR)
C
C         GENERATION OF MODIFIED R
C
       C5=C1/(6.*HX)
       C6=11.*C5
```

```
      C7=C1/(6.*HZ)
      C8=11.*C7
      JJ=0
      N10=NZ - 1
      II=1
       DO 602 I=1,NY
       DO 603 J=1,LY
        JJ= JJ + 1
        IF(JJ.EQ.N10)AK(J,I)=AK(J,I) - C7*W(II,JJ+1)
        IF(JJ.NE.NZ) GO TO 603
         AK(J,I)=AK(J,I) + C8*W(II,JJ)
         JJ=0
         II= II + 1
603    CONTINUE
602    CONTINUE
        N11=(NX - 2)*NZ
        N12=N11 + NZ
         N13=NX - 1
       DO 605 I=1,NY
        MM=I
        DO 604 J=1,NZ
         AK(J,I)=AK(J,I) - C6*U(MM,1)
         J1=NZ + J
         AK(J1,I)=AK(J1,I) + C5*U(MM,1)
         J2=N11 + J
         AK(J2,I)=AK(J2,I) - C5*U(MM,NX)
         J3=N12 + J
         AK(J3,I)=AK(J3,I) + C6*U(MM,NX)
         MM=MM + NY
604     CONTINUE
605     CONTINUE
C
     CALL PRF(YK1,XLL,HY,HMY,NY,NMY,LY,LY1,B,AK,IDIM1,IDIM2
    1,NZ,C1,HX,DR,DR1,DR2,R4)
```

```
C
C       GENERATION OF INITIAL CONDITIONS ON FACE FIVE
C
        II=LZ+1
        JJ=0
         MM=LX + 1
         KK=1
         J1=1
        K3=NZ*NXC
        DO 528 J=1,LY
        JJ=JJ+1
         IF(ID(J).EQ.0)GO TO 30
       AA=C3*(W(II,JJ) + U(MM,KK))
       BB=CC*(EPSY(J,1) + DEPY(J,1))
       V2K(J1)=AA + BB
         J1=J1 + 1
 30      MM=MM + NY
        IF(JJ.NE.NZ) GO TO 528
        JJ=0
         KK=KK + 1
         MM=LX + 1
        II=II+1
 528    CONTINUE
C
C       GENERATTION OF INITIAL CONDITIONS ON FACE TWO
C
        II=LZ + (NY-1)*NX + 1
        JJ=0
         MM=LX + NY
         KK=1
        DO 531 J=1,LY
        JJ=JJ+1
       AA= C3*(W(II,JJ) + U(MM,KK))
       BB=CC*(EPSY(J,NY) + DEPY(J,NY))
```

```
      VNK(I)=AA + BB
        MM=MM + NY
       IF(JJ.NE.NZ) GO TO 531
       JJ=0
        KK=KK + 1
        MM=LX + NY
       II=II + 1
 531   CONTINUE
C
C      CALCULATION FOR UNKNOWN PART OF VECTOR V
C
C      PART OF MATRIX B IS INITIALIZED
C
C
C      CALCULATION OF VARIOUS SUB MATRICE
C
       K1=NZ*NXC
       K2=NZ*(NX-NXC)
       K3=LY + K1
       K4=(NY-1)*NMY + 1
C
        DO 536 I=1,LY
        II=I + LY
        TEMP=0.0
        DO 550 K=1,NFREE
        TEMP=TEMP + B23(I,K)*V2K(K)
 550    CONTINUE
        VNK(I)=VNK(I) - TEMP - B(II,K4)
 536    CONTINUE
C
        DO 538 I=1,LY
        TEMP=0.0
        DO 551 K=1,LY
 551    TEMP=TEMP + BK(I,K)*VNK(K)
```

```
 538     FN(I)=TEMP
C
C
C       FORMULATION OF V1 VECTOR
C
         I2=1
         I1=1
         DO 545 I=1,LY
         IF(ID(I).EQ.0) GO TO 700
         V(I,1)=FN(I1)
         II=I + LY
         V(II,1)=V2K(I1)
         I1=I1 + 1
         GO TO 545
700      II= I + LY
         V(II,1)=FN(NFREE + I2)
         I2=I2 + 1
545      CONTINUE
C
C
         CALL FORMD(XLL,B,NY,NMY,LY1,V,V2K,VNK,IDIM2)
 5000   FORMAT(1X,'THE VALUE OF DETERMINANT IS',E12.4)
 6000    FORMAT(1X,'STEP IS DONE',3X,I4)
        RETURN
        END
```

```
      SUBROUTINE PWDISP(U,V,W,AK,ZK1,XLL,V2K,VNK,L,M,B,ZLOC,FN
     1,EPSZ,DEPZ,EPSZX,DEPZX,EPSYZ,DEPYZ,DR,DR1,DR2)
C
C            IN THIS FILE XLN=ZLN HAS BEEN TAKEN OUT FOR SAVING SPACE
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
      COMMON/BLK2/HX,HY,HZ,HMX,HMY,HMZ,C1,C2,C3,C4
       COMMON/BLK4/IDIM1,IDIM2
         DIMENSION U(LX1,1),V(LY1,1),W(LZ1,1),AK(IDIM1,1),XLL(IDIM2,1)
         DIMENSION VNK(1),L(1),M(1),B(IDIM2,1)
         DIMENSION ZK1(LZ,1),ZLOC(LZ,1),FN(1),V2K(1)
      DIMENSION EPSZ(LZ,1),DEPZ(LZ,1)
      DIMENSION EPSZX(LZ,1),DEPZX(LZ,1),EPSYZ(LY,1),DEPYZ(LY,1)
      DIMENSION DR(IDIM1,1),DR1(IDIM1,1),DR2(IDIM1,1)
C      GENERATION OF R VECTORS
C
C
       CALL FORMR(U,V,NZ,LZ,NX,LX,LY,LX1,LY1,C2,AK,IDIM1)
C
C     ADDITION OF PLASTIC TERMS
C
       CC=2.0*C1
      DO 60 I=1,NZ
      DO 60 J=1,LZ
      AK(J,I)=AK(J,I) + CC*(EPSZ(J,I) + DEPZ(J,I))
60    CONTINUE
C
      CALL SHER(EPSZX,DEPZX,EPSYZ,DEPYZ,LZ,LY,NZ,NX,NY,
     *C1,HX,HY,IDIM1,DR)
C
C       ********* GENERATION OF MODIFIED R
C
         C5=C1/(6.*HX)
```

```
      C6=11.*C5
     C7=C1/(6.*HY)
      C8=11.*C7
     N10=NX -1
     II=1
      JJ=0
     DO 102 I=1,NZ
     DO 103 J=1,LZ
      JJ=JJ + 1
       IF(JJ.EQ.1)AK(J,I)=AK(J,I) - C6*U(II,JJ)
       IF(JJ.EQ.2)AK(J,I)=AK(J,I) + C5*U(II,JJ-1)
       IF(JJ.EQ.N10)AK(J,I)=AK(J,I) - C5*U(II,JJ+1)
       IF(JJ.NE.NX) GO TO 103
       AK(J,I)=AK(J,I) + C6*U(II,JJ)
       JJ=0
        II=II + 1
103    CONTINUE
102   CONTINUE
     DO 104 I=1,NZ
     MM=1
     DO 105 J=1,NXC
     AK(J,I)=AK(J,I) - C8*V(MM,1)
     J1=J + NX
      AK(J1,I)=AK(J1,I) + C7*V(MM,1)
       MM=MM + NZ
105  CONTINUE
104  CONTINUE
       N11=(NY -2)*NX
       N12=N11 + NX
       N13=NY -1
      DO 106 I=1,NZ
       MM=I
      DO 107 J=1,NX
        J2=N11 + J
```

```
                AK(J2,I)=AK(J2,I)  -  C7*V(MM,NY)
               J3=N12 + J
                 AK(J3,I)=AK(J3,I) + C8*V(MM,NY)
                MM= MM + NZ
 107         CONTINUE
 106         CONTINUE
C
C
          CALL PFORM(ZK1,XLL,HZ,HMZ,NZ,NMZ,LZ,LZ1,B,AK,IDIM1,IDIM2,
     *DR,DR1,DR2)
C
C        GENERATION OF INITIAL CONDITION OF FACE SIX
C
C
C        GENERATION OF INITIAL CONDITION ON FACE THREE
C
         II=LX + (NZ-1)*NY + 1
         JJ=0
          MM=LY + NZ
          KK=1
         DO 628 J=1,LZ
         JJ=JJ + 1
          AA=C3*(U(II,JJ) + V(MM,KK))
        BB=CC*(EPSZ(J,NZ) + DEPZ(J,NZ))
        VNK(J)=AA + BB
          MM=MM + NZ
         IF(JJ.NE.NX) GO TO 628
         JJ=0
          KK=KK + 1
          MM=LY + NZ
         II=II + 1
 628     CONTINUE
C
C        CALCULATION OF UNKNOWN PART PF VECTOR W
```

```
C
C       YK1 CONTAINS L4-1
C     INITIALIZATION OF PART OF B TO BE USED AS WORKSPACE
        K1=(NZ-1)*NMZ + 1
         DO 660 I=1,LZ
         II=LZ + I
         VNK(I)=VNK(I) - B(II,K1)
 660     CONTINUE
         DO 670 I=1,LZ
         TEMP=0.0
         II=I + LZ
         DO 671 K=1,LZ
         TEMP=TEMP + ZLOC(I,K)*VNK(K)
 671     CONTINUE
 670     W(II,1)=TEMP
C
         CALL FORMD(XLL,B,NZ,NMZ,LZ1,W,V2K,VNK,IDIM2)
 5000   FORMAT(1X,'THE VALUE OF DETERMINANT IS',3X,E12.4)
 6000   FORMAT(1X,'STEP IS DONE',3X,I4)
        RETURN
        END
```

```
      SUBROUTINE SHER(EPSXY,DEFXY,EPSZX,DEFZX,LX,LZ,NX,NY,NZ,
     1 C1,HY,HZ,IDIM1,DR)
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION EPSXY(LX,1),DEFXY(LX,1),EPSZX(LZ,1),DEFZX(LZ,1)
      DIMENSION DR(IDIM1,1)
CC
      C2=C1/HY
      C3=C1/HZ
      II=1
      DO 10 I=1,NX
      DO 20 J=1,NZ
C
      A1=EPSXY(II,I) + DEFXY(II,I)
      A2=EPSXY(II+1,I) + DEFXY(II+1,I)
      A3=EPSXY(II+2,I) + DEFXY(II+2,I)
C
      DR(II,I)=C2*(-3.*A1 + 4.*A2 - A3)
      II=II + 1
C
      K1=NY - 1
C
      DO 30 K=2,K1
C
      A1=EPSXY(II-1,I) + DEFXY(II-1,I)
      A2=EPSXY(II+1,I) + DEFXY(II+1,I)
C
      DR(II,I)=C2*(-A1 + A2)
      II=II + 1
C
30    CONTINUE
C
      A1=EPSXY(II-2,I) + DEFXY(II-2,I)
      A2=EPSXY(II-1,I) + DEFXY(II-1,I)
```

```
      A3=EPSXY(II,I) + DEPXY(II,I)
C
      DR(II,I)=C2*(A1 - 4.*A2 + 3.*A3)
      II=II + 1
C
20    CONTINUE
      II=1
10    CONTINUE
C
C      ADDITION OF THE SECOND TERM
C
      II=1
      DO 40 I=1,NX
      DO 50 J=1,NY
      K1=(J-1)*NX + I
C
      A1=EPSZX(K1,1) + DEPZX(K1,1)
      A2=EPSZX(K1,2) + DEPZX(K1,2)
      A3=EPSZX(K1,3) + DEPZX(K1,3)
C
      DR(II,I)=DR(II,I) + C3*(-3.*A1 + 4.*A2 - A3)
      II=II+1
C
50    CONTINUE
C
      K2=NZ - 1
C
      DO 60 K=2,K2
      DO 70 J=1,NY
      K1=(J-1)*NX + I
C
      A1=EPSZX(K1,K-1) + DEPZX(K1,K-1)
      A2=EPSZX(K1,K+1) + DEPZX(K1,K+1)
C
```

```
      DR(II,I)=DR(II,I) + C3*(-A1 + A2)
      II=II + 1
C
70    CONTINUE
60    CONTINUE
C
      DO 80 J=1,NY
      K1=(J-1)*NX + I
C
      A1=EPSZX(K1,NZ-2) + DEPZX(K1,NZ-2)
      A2=EPSZX(K1,NZ-1) + DEPZX(K1,NZ-1)
      A3=EPSZX(K1,NZ) + DEPZX(K1,NZ)
C
      DR(II,I)=DR(II,I) + C3*(3.*A1 - 4.*A2 + A3)
      II=II+ 1
C
80    CONTINUE
      II=1
40    CONTINUE
C
      RETURN
      END
         SUBROUTINE PDORM(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2,
     1DR,DR1,DR2)
        IMPLICIT REAL*8(A-H,O-W),REAL*16(X),REAL*8(Y,Z)
         DIMENSION XK1(LX,1),XLL(LX1,1),AK(IDIM1,1),B(IDIM2,1)
       DIMENSION DR(IDIM1,1),DR1(IDIM1,1),DR2(IDIM1,1)
C
C          THIS SUBROUTINE IS FORMED USING A PART OF OLD SUBROUTINE
C         ALL THE ARGUMENTS ARE REPLACED BY RESPECTIVE VALUES FOR Y AND
C
C
C       INTERPOLATION OF VALUES
C
```

```
      X0=0.0
      X1=X0 + HX
      JJ=3
      X=X0
      K1=NX - 2
      K2=NMX - 1
      DO 80 J=1,K1
      DO 90 I=1,LX
      B(I,1)=AK(I,J)
      B(I,2)=(AK(I,J+1) - AK(I,J))/HX
      B(I,3)=(AK(I,J+2)-AK(I,J+1)-B(I,2)*HX)/(2.*HX*HX)
     KK=I+LX
     B(KK,1)=DR(I,J)
     B(KK,2)=(DR(I,J+1)-DR(I,J))/HX
     B(KK,3)=(DR(I,J+2)-DR(I,J+1)-B(KK,2)*HX)/(2.*HX*HX)
 90   CONTINUE
      DO 101 K=1,K2
      X=X + HMX
      AA=X - X0
      BB=X - X1
      JJ=JJ + 1
      DO 101 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
C
     KK=I+LX
     DR1(I,JJ-3)=B(KK,1)+AA*B(KK,2)+AA*BB*B(KK,3)
C
 101  CONTINUE
      X0=X1
      X1=X1 + HX
      X=X0
 80   CONTINUE
         X=X0 + HX
C
C     INTERPOLATION FOR LAST INTERVAL
```

```
      DO 110 K=1,K2
      X=X + HMX
      JJ=JJ + 1
      AA=X - X0
      BB=X - X1
      DO 110 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
C
      KK=I+LX
      DR1(I,JJ-3)=B(KK,1)+AA*B(KK,2)+AA*BB*B(KK,3)
C
 110  CONTINUE
C
C     GENERATION OF RI MATRIX INCLUSING MINOR AND MAJOR NODES
C
      K1=(NX-1)*NMX + 1
      DO 120 I=1,LX
      B(I,K1)=AK(I,1)
      DR2(I,1)=DR(I,1)
 120  CONTINUE
      JJ=3
      JB=0
      JC=1
      DO 130 J=2,NX
      DO 140 JA=1,K2
      K1=K1 + 1
      JJ=JJ + 1
      JB=JB+1
      JC=JC+1
      DO 150 I=1,LX
      B(I,K1)=B(I,JJ)
      DR2(I,JC)=DR1(I,JB)
 150  CONTINUE
```

```
 140    CONTINUE
        K1=K1 + 1
       JC=JC+1
        DO 160 I=1,LX
        B(I,K1)=AK(I,J)
       DR2(I,JC)=DR(I,J)
 160    CONTINUE
 130    CONTINUE
 8000   FORMAT(1X,'THE RX VECTORS ARE'/)
C       CALCULATION OF RI-RI-1
        K2=(NX-1)*NMX
        DO 170 J=1,K2
        K3=K2 + J
        K4=K3 + 1
        DO 180 I=1,LX
       B(I,K3)=B(I,K4) - B(I,K3) + (DR2(I,J) +  DR2(I,J+1))*HMX/2.
 180    CONTINUE
 170    CONTINUE
        ITER=ITER + 1
C       INITIALIZATION OF B
C       CALCULATION OF M
C
         DO 201 I=1,LX
         II=I + LX
         DO 201 J=1,K2
         JJ=K2 + J
         TEMP=0.0
         DO 202 K=1,LX
         TEMP=TEMP + XK1(I,K)*B(K,JJ)
 202     CONTINUE
         B(I,J)=HMX*TEMP/2.
         B(II,J)=TEMP
 201     CONTINUE
C
C       CALCULATION OF FN
```

```
C
      K3=K2 + 1
      K4=K3 + 1
      DO 230 I=1,LX1
 230  B(I,K3)=0.0
      DO 240 J=1,K2
      DO 250 I=1,LX1
       TEMP=0.0
       DO 251 K=1,LX1
       TEMP=TEMP + XLL(K,I)*B(K,K3)
 251   CONTINUE
 250   B(I,K4)=TEMP
      DO 260 I=1,LX1
      B(I,K3)=B(I,K4) + B(I,J)
 260  CONTINUE
 240  CONTINUE
 3    FORMAT(1X,'THE FN VECTOR IS GIVEN'//)
 7000 FORMAT(1X,10E12.4)
       RETURN
      END

       SUBROUTINE PRF(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2,NZ
     1,C1,H,DR,DR1,DR2,AINCR)
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION XK1(LX,1),XLL(IDIM2,1),AK(IDIM1,1),B(IDIM2,1)
      DIMENSION DR(IDIM1,1),DR1(IDIM1,1),DR2(IDIM1,1)
C     DIMEN XLL HAS BEEN CHANGED TO ACCOM. LY NE.LZ
C
C         THIS SUBROUTINE IS FORMED USING A PART OF OLD SUBROUTINE
C        ALL THE ARGUMENTS ARE REPLACED BY RESPECTIVE VALUES FOR Y AND
C
C
C      INTERPOLATION OF VALUES
C
      XO=0.0
```

```
      X1=XO + HX
      JJ=3
      X=XO
      K1=NX - 2
      K2=NMX - 1
      DO 80 J=1,K1
      DO 90 I=1,LX
      B(`.1)=AK(I,J)
      B(I,2)=(AK(I,J+1) - AK(I,J))/HX
      B(I,3)=(AK(I,J+2)-AK(I,J+1)-B(I,2)*HX)/(2.*HX*HX)
C
     KK=I + LX
     B(KK,1)=DR(I,J)
     B(KK,2)=(DR(I,J+1)-DR(I,J))/HX
     B(KK,3)=(DR(I,J+2)-DR(I,J+1)-B(KK,2)*HX)/(2.*HX*HX)
 90   CONTINUE
      DO 101 K=1,K2
      X=X + HMX
      AA=X - XO
      BB=X - X1
       JJ=JJ + 1
       DO 101 I=1,LX
       B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
C      KK=I + LX
       DR1(I,JJ-3)=B(KK,1)+AA*B(KK,2)+AA*BB*B(KK,3)
C
 101  CONTINUE
      XO=X1
      X1=X1 + HX
      X=XO
 80   CONTINUE
         X=XO + HX
C
C     INTERPOLATION FOR LAST INTERVAL
```

```
      DO 110 K=1,K2
      X=X + HMX
      JJ=JJ + 1
      AA=X - X0
      BB=X - X1
      DO 110 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
C
      KK=I + LX
      DR1(I,JJ-3)=B(KK,1)+AA*B(KK,2)+AA*BB*B(KK,3)
C
  110 CONTINUE
C
C     GENERATION OF RI MATRIX INCLUSING MINOR AND MAJOR NODES
C
      K1=(NX-1)*NMX + 1
      DO 120 I=1,LX
      B(I,K1)=AK(I,1)
      DR2(I,1)=DR(I,1)
  120 CONTINUE
      JJ=3
C
      JB=0
      JC=1
C
      DO 130 J=2,NX
      DO 140 JA=1,K2
      K1=K1 + 1
      JJ=JJ + 1
      JB=JB+1
      JC=JC+1
      DO 150 I=1,LX
      B(I,K1)=B(I,JJ)
      DR2(I,JC)=DR1(I,JB)
```

```
 150    CONTINUE
 140    CONTINUE
        K1=K1 + 1
       JC=JC+1
        DO 160 I=1,LX
        B(I,K1)=AK(I,J)
       DR2(I,JC)=DR(I,J)
 160    CONTINUE
 130    CONTINUE
 8000   FORMAT(1X,'THE RX VECTORS ARE'/)
C       CALCULATION OF RI-RI-1
C
       A2=AINCR*HMX*C1/(12.0*H)
       A1=-11.0*A2
       N10=2*NZ
       SI=HMX/((NX - 1)*HX)
C
        K2=(NX-1)*NMX
        DO 170 J=1,K2
C
       DY1=(J -1)*SI
       DY2=DY1 + SI
       AL1=4.*DY1*(1 - DY1)
       AL2=4.*DY2*(1 - DY2)
       A11=A1*(AL1 + AL2)
       A22=A2*(AL1 + AL2)
        K3=K2 + J
        K4=K3 + 1
        DO 180 I=1,LX
       IF(I.GT.NZ) GO TO 181
       B(I,K3)=B(I,K4) - B(I,K3) + (DR2(I,J) + DR2(I,J+1))*HMX/2.+A11
       GO TO 180
 181   IF(I.GT.N10)GO TO 182
       B(I,K3)=B(I,K4) - B(I,K3) + (DR2(I,J) + DR2(I,J+1))*HMX/2. + A22
```

```
      GO TO 180
182   B(I,K3)=B(I,K4) - B(I,K3) + (DR2(I,J) + DR2(I,J+1))*HMX/2.
 180   CONTINUE
 170   CONTINUE
       ITER=ITER + 1
C      INITIALIZATION OF B
C      CALCULATION OF M
C
        DO 201 I=1,LX
        II=I + LX
        DO 201 J=1,K2
        JJ=K2 + J
        TEMP=0.0
        DO 202 K=1,LX
        TEMP=TEMP + XK1(I,K)*B(K,JJ)
 202    CONTINUE
        B(I,J)=HMX*TEMP/2.
        B(II,J)=TEMP
 201    CONTINUE
C
C      CALCULATION OF FN
C
       K3=K2 + 1
       K4=K3 + 1
       DO 230 I=1,LX1
 230   B(I,K3)=0.0
       DO 240 J=1,K2
       DO 250 I=1,LX1
        TEMP=0.0
        DO 251 K=1,LX1
        TEMP=TEMP + XLL(K,I)*B(K,K3)
 251    CONTINUE
 250    B(I,K4)=TEMP
       DO 260 I=1,LX1
```

```
      B(I,K3)=B(I,K4) + B(I,J)
260   CONTINUE
240   CONTINUE
3     FORMAT(1X,'THE FN VECTOR IS GIVEN'//)
7000  FORMAT(1X,10E12.4)
       RETURN
      END
        SUBROUTINE PFORM(XK1,XLL,HX,HMX,NX,NMX,LX,LX1,B,AK,IDIM1,IDIM2,
     1DR,DR1,DR2)
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION XK1(LX,1),XLL(IDIM2,1),AK(IDIM1,1),B(IDIM2,1)
      DIMENSION DR(IDIM1,1),DR1(IDIM1,1),DR2(IDIM1,1)
C     DIMEN XLL HAS BEEN CHANGED TO ACCOM. LY NE.LZ
C
C          THIS SUBROUTINE IS FORMED USING A PART OF OLD SEBROUTINE
C         ALL THE ARGUMENTS ARE REPLACED BY RESPECTIVE VALUES FOR Y AND
C
C
C       INTERPOLATION OF VALUES
C
       X0=0.0
       X1=X0 + HX
       JJ=3
       X=X0
       K1=NX - 2
       K2=NMX - 1
       DO 80 J=1,K1
       DO 90 I=1,LX
       B(I,1)=AK(I,J)
       B(I,2)=(AK(I,J+1) - AK(I,J))/HX
       B(I,3)=(AK(I,J+2)-AK(I,J+1)-B(I,2)*HX)/(2.*HX*HX)
C
      KK=I + LX
      B(KK,1)=DR(I,J)
```

```
      B(KK,2)=(DR(I,J+1)-DR(I,J))/HX
      B(KK,3)=(DR(I,J+2)-DR(I,J+1)-B(KK,2)*HX)/(2.*HX*HX)
 90   CONTINUE
      DO 101 K=1,K2
      X=X + HMX
      AA=X - XO
      BB=X - X1
      JJ=JJ + 1
      DO 101 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
C     KK=I + LX
      DR1(I,JJ-3)=B(KK,1)+AA*B(KK,2)+AA*BB*B(KK,3)
C
 101  CONTINUE
      XO=X1
      X1=X1 + HX
      X=XO
 80   CONTINUE
         X=XO + HX
C
C     INTERPOLATION FOR LAST INTERVAL
      DO 110 K=1,K2
      X=X + HMX
      JJ=JJ + 1
      AA=X - XO
      BB=X - X1
      DO 110 I=1,LX
      B(I,JJ)=B(I,1) + AA*B(I,2) + AA*BB*B(I,3)
C
      KK=I + LX
      DR1(I,JJ-3)=B(KK,1)+AA*B(KK,2)+AA*BB*B(KK,3)
C
 110  CONTINUE
C
```

```
C       GENERATION OF RI MATRIX INCLUSING MINOR AND MAJOR NODES
C
        K1=(NX-1)*NMX + 1
        DO 120 I=1,LX
        B(I,K1)=AK(I,1)
        DR2(I,1)=DR(I,1)
 120    CONTINUE

         JJ=3
C
       JB=0
       JC=1
C
         DO 130 J=2,NX
         DO 140 JA=1,K2
         K1=K1 + 1
         JJ=JJ + 1
       JB=JB+1
       JC=JC+1
         DO 150 I=1,LX
         B(I,K1)=B(I,JJ)
       DR2(I,JC)=DR1(I,JB)
 150     CONTINUE
 140     CONTINUE
         K1=K1 + 1
       JC=JC+1
         DO 160 I=1,LX
         B(I,K1)=AK(I,J)
       DR2(I,JC)=DR(I,J)
 160     CONTINUE
 130     CONTINUE
 8000    FORMAT(1X,'THE RX VECTORS ARE'/)
C        CALCULATION OF RI-RI-1
         K2=(NX-1)*NMX
         DO 170 J=1,K2
```

```
      K3=K2 + J
      K4=K3 + 1
      DO 180 I=1,LX
     B(I,K3)=B(I,K4) - B(I,K3) + (DR2(I,J) + DR2(I,J+1))*HMX/2.
 180  CONTINUE
 170  CONTINUE
      ITER=ITER + 1
C
C     INITIALIZATION OF B
C     CALCULATION OF M
C
       DO 201 I=1,LX
       II=I + LX
       DO 201 J=1,K2
       JJ=K2 + J
       TEMP=0.0
       DO 202 K=1,LX
       TEMP=TEMP + XK1(I,K)*B(K,JJ)
 202   CONTINUE
       B(I,J)=HMX*TEMP/2.
       B(II,J)=TEMP
 201   CONTINUE
C
C     CALCULATION OF FN
C
      K3=K2 + 1
      K4=K3 + 1
      DO 230 I=1,LX1
 230  B(I,K3)=0.0
      DO 240 J=1,K2
      DO 250 I=1,LX1
       TEMP=0.0
       DO 251 K=1,LX1
       TEMP=TEMP + XLL(K,I)*B(K,K3)
 251   CONTINUE
```

```
250      B(I,K4)=TEMP
        DO 260 I=1,LX1
        B(I,K3)=B(I,K4) + B(I,J)
260     CONTINUE
240     CONTINUE
3       FORMAT(1X,'THE FN VECTOR IS GIVEN'//)
7)00    FORMAT(1X,10E12.4)
         RETURN
        END
        SUBROUTINE SHEAXY(U,V,LX1,LY1,HY,HX,PR,IDIM1,AK,NX,NY,NZ,NCODE)
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION U(LX1,1),V(LY1,1),AK(IDIM1,1)
C
C        NCODE=0  SHEARING STRAIN
C        MCODE=1  SHEARING STRESSS
C        SUBROUTINE TO CALCULATE SHEAR STRESS
C
        C1=0.25
        C2=C1/HY
        C3=C1/HX
        II=1
C
        DO 10 I=1,NX
        DO 20 J=1,NZ
        IF(I.GT.1) GO TO 21
        AK(II,I)=C2*(-3.*U(II,I) + 4.*U(II+1,I) - U(II+2,I))
        GO TO 22
21      AK(II,I)=0.0
22      II=II + 1
        K1=NY - 1
        DO 30 K=2,K1
        IF(I.EQ.NX) GO TO 31
        AK(II,I)=C2*(-U(II-1,I) + U(II+1,I))
        GO TO 32
```

```
31    AK(II,I)=0.0
32    II=II+ 1
30    CONTINUE
      IF(I.GT.1)GO TO 33
      AK(II,I)=C2*(U(II-2,I) - 4.*U(II-1,I) + 3.*U(II,I))
      GO TO 34
33    AK(II,I)=0.0
34    II=II+1
20    CONTINUE
      II=1
10    CONTINUE
C
C      THE SECOND PART ADDED TO THE PREVIOUS
C
      II=0
C
      DO 40 I=1,NZ
      DO 50 J=1,NY
C
      A=C3*(-3.*V(I,J) + 4.*V(I+NZ,J) - V(I+2*NZ,J))
      II=II + 1
      AK(II,1)=AK(II,1) + A
50    CONTINUE
40    CONTINUE
C
      II=1
C
      KK=1
C
      K1=NX - 1
      DO 60 I=2,K1
      DO 70 J=1,NZ
      DO 80 K=1,NY
      IF(K.EQ.1.OR.K.EQ.NY) GO TO 81
```

```
      A=C3*(-V(KK,K) + V(KK+2*NZ,K))
      AK(II,I)=AK(II,I) + A
81    II=II + 1
80    CONTINUE
C
      KK=KK + 1
C
70    CONTINUE
      II=1
60    CONTINUE
C
      IF(NCODE.EQ.0) RETURN
      NG1=IDIM1*NX
A      A=1.0/(1.0 + PR)
      CALL SCAVE1(AK,A,NG1)
      RETURN
      END
      SUBROUTINE SHEAYZ(U,V,LX1,LY1,HY,HX,PR,IDIM1,AK,NX,NY,NZ,NCODE)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION U(LX1,1),V(LY1,1),AK(IDIM1,1)
C
C      NCODE=0  SHEARING STRAIN
C      NCODE=1  SHEARING STRESSS
C      SUBROUTINE TO CALCULATE SHEAR STRESS
C
      C1=0.25
      C2=C1/HY
      C3=C1/HX
      II=1
C
      DO 10 I=1,NX
      DO 20 J=1,NZ
      AK(II,I)=0.0
      II=II + 1
```

```
      K1=NY - 1
      DO 30 K=2,K1
      IF(I.EQ.1.OR.I.EQ.NX) GO TO 31
      AK(II,I)=C2*(-U(II-1,I) + U(II+1,I))
      GO TO 32
31      AK(II,I)=0.0
32    II=II+ 1
30    CONTINUE
      AK(II,I)=0.0
      II=II + 1
20    CONTINUE
      II=1
10    CONTINUE
C
C      THE SECOND PART ADDED TO THE PREVIOUS
C
C

      II=1
C
      KK=1
C
      K1=NX - 1
      DO 60 I=2,K1
      DO 70 J=1,NZ
      DO 80 K=1,NY
      IF(K.EQ.1.OR.K.EQ.NY) GO TO 81
      A=C3*(-V(KK,K) + V(KK+2*NZ,K))
      AK(II,I)=AK(II,I) + A
81    II=II+ 1
80    CONTINUE
C
      KK=KK + 1
C
70    CONTINUE
```

```
      II=1
60    CONTINUE
C
      IF(NCODE.EQ.0) RETURN
      NG1=IDIM1*NX
4      A=1.0/(1.0 + PR)
      CALL SCAVE1(AK,A,NG1)
      RETURN
      END
       SUBROUTINE STRESS(A,B,C,L1,L2,L3,M1,M2,M3,PR,TRESS,N1,N2,N3)
          IMPLICIT REAL*8(A-H,O-Z)
          DIMENSION TRESS(L1,1),A(N1,1),B(N2,1),C(N3,1)
 1000           FORMAT(5X,F12.6)
C
C         L1 IS THE NUMBER OF ROWS IN A MATRIX
C         L2 IS THE NUMBER OF ROWS IN B MATRIX
C         L3 IS THE NUMBER OF ROWS IN C MATRIX
C          M1 IS THE NUMBER OF COLUMNS IN A MATRIX
C           M2 IS THE NUMBER OF COLUMNS IN B MATRIX
C           M3 IS THE NUMBER OF COLUMNS IN C MATRIX
C
C           FORMULATION OF STRESS MATRIX
C
          D=(1 + PR)*(1-2*PR)
          D=1/D
          DD=(1 - PR)
          II= L2 + 1
          JJ=0
           DO 10 I=1,M1
           MM=L3 + I
           KK=1
           DO 20 J=1,L1
            JJ=JJ + 1
            LL=L1 + J
```

```
            TRESS(J,I)=D*(DD*A(LL,I) + PR*B(II,JJ) + PR*C(MM,KK))
            MM=MM + M1
             IF(JJ.NE.M2) GO TO 20
            KK=KK + 1
            MM=L3 + I
            II=II + 1
            JJ=0
 20         CONTINUE
 10         CONTINUE
             RETURN
             END
C
C        SUBROUTINE FOR INITIALIZATION,CONSTANT MULTIPLICATION
C        AND ADDITION OF TWO MATRICES
C
      SUBROUTINE MATIN(A,N)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),B(1)
C
      DO 10 I=1,N
10    A(I)=0.0
      RETURN
C
C      C(M,N)=R*C(M,N)
      ENTRY SCAVE1(A,R,N)
      DO 20 I=1,N
20    A(I)=R*A(I)
      RETURN
C
C    D(M,N)=R*C(M,N)
C
      ENTRY SCAVE2(A,R,N,B)
      DO 30 I=1,N
30    B(I)=R*A(I)
```

```
      RETURN
C
C    B(M,N)=A(M,N)+B(M,N)
C
      ENTRY MATADD(A,N,B)
      DO 40 I=1,N
40    B(I)=B(I) + A(I)
      RETURN
C
C     B(I)=A(I)
C
      ENTRY MATEQU(A,N,B)
      DO 50 I=1,N
50    B(I)=A(I)
      RETURN
      END
        SUBROUTINE OUTPUT(A,N,M,IHEAD,N1,N2)
        IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION A(N,1)
       IF(IHEAD.EQ.1) GO TO 10
       IF(IHEAD.EQ.2) GO TO 20
       WRITE(6,1000)
 1000    FORMAT(1X,'THE W-DOT MATRIX '//)
       GO TO 30
 10    WRITE(6,2000)
 2000    FORMAT(1X,'THE U-DOT MATRIX'//)
       GO TO 30
 20    WRITE(6,3000)
 3000    FORMAT(1X,'THE V-DOT MATRIX'//)
 30      DO 40 I=N1,N2
          WRITE(6,4000)(A(I,J),J=1,M)
 40    CONTINUE
 4000   FORMAT(1X,10F12.6)
 5000  FORMAT(1X//)
       RETURN
       END
```

```
      SUBROUTINE PLSTRN(EET,DELEP,EPX,EPY,EPZ,EPXY,EPYZ,EPZX,
     *DEPX,DEPY,DEPZ,DEPXY,DEPYZ,DEPZX)
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
C
      DIMENSION EET(NY,NZ,1),DELEP(NY,NZ,1),EPX(LX,1),EPY(LY,1)
      DIMENSION EPZ(LZ,1)
      DIMENSION EPXY(LX,1),EPYZ(LY,1),EPZX(LZ,1),DEPX(LX,1),DEPY(LY,1)
      DIMENSION DEPZ(LZ,1),DEPXY(LX,1),DEPYZ(LY,1),DEPZX(LZ,1)
C
C      CALCULATION OF INCREMENTAL PLASTIC STRAIN
C
      DO 10 I=1,NX
      DO 10 J=1,NZ
      DO 10 K=1,NY
C
      IF(DELEP(K,J,I).LE.0.0)GO TO 10
1000  FORMAT(3I5,F10.5)
C
      I1=(J-1)*NY + K
      I2=(I-1)*NZ + J
      I3=(K-1)*NX + I
C
      A1=DELEP(K,J,I)/EET(K,J,I)
      DEPX(I1,I)=A1*(2.0*EPX(I1,I) - EPY(I2,K) - EPZ(I3,J))/3.0
      DEPXY(I1,I)=A1*EPXY(I1,I)
C
      DEPY(I2,K)=A1*(2.*EPY(I2,K) - EPZ(I3,J) - EPX(I1,I))/3.0
      DEPYZ(I2,K)=A1*EPYZ(I2,K)
C
      DEPZ(I3,J)=A1*(2.0*EPZ(I3,J) - EPX(I1,I) - EPY(I2,K))/3.0
      DEPZX(I3,J)=A1*EPZX(I3,J)
1001  FORMAT(5X,3I5,5X,'DEPX=',F10.5,5X,'DEPY= [illegible]).5
```

```
     *,5X,'DEPZ=',F10.5)
C
10      CONTINUE
        RETURN
        END
         SUBROUTINE EFTRES(AK,AK1,AK2,AK3,AK4,AK5,ESTRES,NCODE)
         IMPLICIT REAL*8(A-H,O-Z)
         COMMON/BLK1/NX,NY,NZ,NXC,LX,LY,LZ,LX1,LY1,LZ1,NMX,NMY,NMZ
         DIMENSION AK(LX,1),AK1(LY,1),AK2(LZ,1)
         DIMENSION AK3(LX,1),AK4(LY,1),AK5(LZ,1)
         DIMENSION ESTRES(NY,NZ,1)
C
C         NCODE=0 EFFECTIVE STRAIN
C         NCODE=1 EFFECTIVE STRESS
C         SUBROUTINE TO CALCULATE EFFECTIVE STRESS
C
         DO 10 I=1,NX
         DO 10 J=1,NZ
         DO 10 K=1,NY
C
         C1=AK((J-1)*NY + K,I)
         C2=AK3((J-1)*NY + K,I)
         C3=AK1((I-1)*NZ + J,K)
         C4=AK4((I-1)*NZ + J,K)
         C5=AK2((K-1)*NX + I,J)
         C6=AK5((K-1)*NX +I,J)
C
         C12=6.*(C2*C2 + C4*C4 + C6*C6)
C
         C7=(C1-C3)**2
         C8=(C3-C5)**2
         C9=(C5-C1)**2
C
         C11= DSQRT(C7 + C8 + C9 + C12)
```

```
C
      ESTRES(K,J,I)=C11
10    CONTINUE
C
      IF(NCODE.EQ.1) GO TO 20
      A=DSQRT(2.D0)/3.0
      NG1=NX*LX
      CALL SCAVE1(ESTRES,A,NG1)
      RETURN
20    A=1./DSQRT(2.D0)
      NG1=NX*LX
      CALL SCAVE1(ESTRES,A,NG1)
      RETURN
      END
      SUBROUTINE EH(TT,HE)
      IMPLICIT REAL*8(A-H,O-Z)
C
      IF(TT.GE.1.0.AND.TT.LT.1.2) GO TO 1
      IF(TT.GE.1.2.AND.TT.LT.1.28) GO TO 2
      IF(TT.GE.1.28.AND.TT.LT.1.36) GO TO 3
      IF(TT.GE.1.36.AND.TT.LT.1.44) GO TO 4
      IF(TT.GE.1.44.AND.TT.LT.1.48) GO TO 5
      IF(TT.GE.1.48.AND.TT.LT.1.528) GO TO 6
      IF(TT.GE.1.528.AND.TT.LT.1.56) GO TO 7
      IF(TT.GE.1.56.AND.TT.LT.1.60) GO TO 8
      IF(TT.GE.1.60.AND.TT.LT.1.64) GO TO 9
      IF(TT.GE.1.64.AND.TT.LT.1.692) GO TO 10
      IF(TT.GE.1.692.AND.TT.LT.1.744) GO TO 11
      IF(TT.GE.1.744.AND.TT.LT.1.832) GO TO 12
      IF(TT.GE.1.832.AND.TT.LT.1.868) GO TO 13
      IF(TT.GE.1.868.AND.TT.LT.2.296) GO TO 14
      IF(TT.SE.2.296.AND.TT.LT.2.500) GO TO 15
      IF(TT.GE.2.5.AND.TT.LT.2.768) GO TO 16
      IF(TT.GE.2.768.AND.TT.LT.2.984) GO TO 17
```

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```
      IF(TT.GE.2.984.AND.TT.LT.3.142) GO TO 18
      IF(TT.GE.3.142.AND.TT.LT.3.264) GO TO 19
      IF(TT.GE.3.264.AND.TT.LT.3.364) GO TO 20
      IF(TT.GE.3.364.AND.TT.LT.3.436) GO TO 21
      IF(TT.GE.3.436.AND.TT.LT.3.488) GO TO 22
      IF(TT.GE.3.488) HE=.000115
      RETURN
1      HE=5.4642
      RETURN
2      HE=3.98
      RETURN
3      HE=.7843
      RETURN
4      HE=.3231
      RETURN
5      HE=.21333
      RETURN
6      HE=.1519
      RETURN
7      HE=.11536
      RETURN
8      HE=.10565
      RETURN
9      HE=.079051
      RETURN
10     HE=.06473
      RETURN
11     HE=.06061
      RETURN
12      HE=.05081
      RETURN
13     HE=.04119
      RETURN
14     HE=.02309
```

```
      RETURN
15      HE=.01817
      RETURN
16     HE=.01391
      RETURN
17     HE=.01084
      RETURN
18     HE=.007666
      RETURN
19     HE=.005727
      RETURN
20     HE=.004542
      RETURN
21       HE=.003163

      RETURN
22      HE=.00221
      RETURN
      END
```